

MEMBERSHIP PROBLEM IN GROUPS ACTING FREELY ON \mathbb{Z}^n -TREES

ANDREY NIKOLAEV AND DENIS SERBIN

ABSTRACT. Groups acting freely on \mathbb{Z}^n -trees (\mathbb{Z}^n -free groups) play a key role in the study of non-archimedean group actions. Following Stallings' ideas, we develop graph-theoretic techniques to investigate subgroup structure of \mathbb{Z}^n -free groups. As an immediate application of the presented method, we give an effective solution to the Uniform Membership Problem and the Power Problem in \mathbb{Z}^n -free groups.

CONTENTS

1. Introduction	2
1.1. The Membership Problems in groups	2
1.2. \mathbb{Z}^n -free groups	3
2. Preliminaries	4
2.1. Lyndon length functions and free actions on trees	4
2.2. Regular length functions	5
2.3. Infinite words and length functions	6
2.4. Groups with free length functions in \mathbb{Z}^n	8
2.5. Groups with free regular length functions in \mathbb{Z}^n , a refinement	10
2.6. Normal forms in G_{i+1}	13
2.7. Normal forms in G	18
3. $\mathcal{B}(G)$ -graphs	19
3.1. Labeled graphs	19
3.2. Free foldings	21
3.3. C -components	22
3.4. Reduced C -components	23
3.5. ST_m -complexes	24
4. Folded $\mathcal{B}(G)$ -graphs	28
4.1. Weakly folded $\mathcal{B}(G)$ -graphs	29
4.2. Induction step for $\mathbf{E}(\cdot)$	31
4.3. Induction step for $\mathbf{C}(\cdot)$	32
4.4. Induction step for $\mathbf{R}(\cdot)$ and $\mathbf{R}^+(\cdot)$	37
5. Results and conclusions	38

5.1. Note on effectiveness	38
5.2. Solutions to the Subgroup Membership and Power Problems	39
References	40

1. INTRODUCTION

The goal of the present work is to develop combinatorial techniques for studying algorithmic problems in groups acting freely on \mathbb{Z}^n -trees, most notably The Uniform Membership Problem.

1.1. The Membership Problems in groups. Recall that a finitely generated group

$$G = \langle x_1, x_2, \dots, x_k \mid r_1, r_2, \dots \rangle$$

is said to have *solvable membership problem* (or *solvable uniform membership problem*) if there is an algorithm which, for any finite family of words u, w_1, w_2, \dots, w_n in $\{x_1, x_2, \dots, x_k\}^{\pm 1}$ decides whether or not the element of G represented by u belongs to the subgroup of G generated by the elements of G corresponding to w_1, w_2, \dots, w_n (this definition does not depend on the choice of a finite generating set for G). Similarly, if $H \leq G$ is a specific subgroup, then H is said to have solvable membership problem in G if there is an algorithm deciding for any word u in $\{x_1, x_2, \dots, x_k\}^{\pm 1}$ whether u represents an element of H .

The membership problem in groups (and subgroup structure in general) has been extensively studied for many classes of groups, employing multitude of different techniques and ideas. It is outside the scope of this paper to give a comprehensive account of results in this area, but we would like to point out certain developments that are of particular interest to us.

The framework for our study comes from the seminal paper [20] by J. Stallings, where he introduced an extremely useful notion of a folding of graphs and initiated the study of subgroups (and automorphisms) of free groups via folded directed labeled graphs. This approach turned out to be very influential, it allowed researchers to prove many new results and simplify old proofs. Detailed exposition of these results can be found, for example, in [7].

Later this method was used by A. Myasnikov, V. Remeslennikov, and D. Serbin who showed that elements of $F^{\mathbb{Z}[t]}$ (and, hence, of all its subgroups which are fully residually free groups) can be viewed as reduced infinite words over the generating set of F (see [16]). Then it turned out that many algorithmic problems for finitely generated fully

residually free groups can be solved by essentially the same methods as in standard free groups. Indeed, in [17] an analog of Stallings' foldings was introduced for an arbitrary finitely generated subgroup of $F^{\mathbb{Z}[t]}$, which allows one to solve effectively the membership problem in $F^{\mathbb{Z}[t]}$, as well as in its arbitrary finitely generated subgroup. Next, in [9] this technique was further developed to obtain a solution to many algorithmic problems, such as the intersection problem, the conjugacy problem, the malnormality problem etc. In [18] this technique was used to obtain an algorithmic solution to the finite index problem and several other results on subgroup structure of fully residually free groups.

Groups acting freely on \mathbb{Z}^n -trees (\mathbb{Z}^n -free groups) generalize fully residually free groups, and in the present work we attempt to initiate the investigation of their subgroup structure employing graph-theoretic techniques similar to those mentioned above.

Note that \mathbb{Z}^n -free groups are hyperbolic relative to maximal abelian subgroups. The subgroup membership problem is in general undecidable in hyperbolic groups, as shown by Rips [19]. One notable point is that in our study we present a large subclass of relatively hyperbolic groups for which the membership problem is decidable.

The authors are extremely grateful to Alexei Myasnikov, Olga Kharlampovich and Volker Diekert for insightful discussions and many helpful comments and suggestions.

1.2. \mathbb{Z}^n -free groups. The class of groups acting freely on \mathbb{Z}^n -trees became very important in the study of actions on Λ -trees, where Λ is an arbitrary ordered abelian group. It turned out that group actions on \mathbb{Z}^n -trees play a key role in building a unified theory of non-Archimedean actions, length functions and infinite words. The natural effectiveness of all constructions (which is not the case for \mathbb{R} -trees) came along with a robust algorithmic theory.

For detailed exhibition of the context of \mathbb{Z}^n -free groups we refer the reader to [10]. Here we limit ourselves to noting that groups with regular \mathbb{Z}^n -free action play a key role in recent advances (through the effort of Chiswell and Müller [4], and Miasnikov, Kharlampovich, Remeslenikov and Serbin [10, 11]) towards the solution on the following long-standing problem of Alperin–Bass in the non-archimedean case.

Problem. *Describe finitely presented (finitely generated) groups acting freely on an arbitrary Λ -tree.*

Characterization of finitely generated groups with free regular length functions in \mathbb{Z}^n given in [10] is a departure point of our present work,

and we proceed by introducing the necessary terminology, citing relevant results and inspecting class of finitely generated groups with free regular length functions in \mathbb{Z}^n in more detail.

2. PRELIMINARIES

The notions introduced below can be found in a number of other works, for example, in [11, 10]. However, to keep the present exposition reasonably self-contained we choose to include a number of definitions and statements here.

2.1. Lyndon length functions and free actions on trees. Let G be a group and A an ordered abelian group. Then a function $l : G \rightarrow A$ is called a (*Lyndon*) *length function* on G if the following conditions hold:

- (L1) $\forall g \in G : l(g) \geq 0$ and $l(1) = 0$,
- (L2) $\forall g \in G : l(g) = l(g^{-1})$,
- (L3) $\forall g, f, h \in G : c(g, f) > c(g, h) \rightarrow c(g, h) = c(f, h)$,
where $c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f))$.

Notice that $c(g, f)$ may not be defined in A (if $l(g) + l(f) - l(g^{-1}f)$ is not divisible by 2), so in the axiom (L3) we assume that A is canonically embedded into a divisible ordered abelian group $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [16] for details).

It is not difficult to derive the following two properties of length functions from the axioms (L1)–(L3):

- $\forall g, f \in G : l(gf) \leq l(g) + l(f)$,
- $\forall g, f \in G : 0 \leq c(g, f) \leq \min\{l(g), l(f)\}$.

A length function $l : G \rightarrow A$ is called *free* if it satisfies the following two axioms.

- (L4) $\forall g \in G : c(g, f) \in A$,
- (L5) $\forall g \in G : g \neq 1 \rightarrow l(g^2) > l(g)$.

For elements $g_1, \dots, g_n \in G$ we write

$$g = g_1 \circ \dots \circ g_n$$

if $g = g_1 \cdots g_n$ and $l(g) = l(g_1) + \dots + l(g_n)$.

An *A-metric space* is a pair (X, d) , where $d : X \times X \rightarrow A$ is a function such that

- (a) $d(x, y) = 0 \iff x = y$,
- (b) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

In this event d is called *distance function*. A basic example of an *A-metric space* is A itself with the usual distance function. An *isometry* between *A-metric spaces* is a map that preserves distance.

Recall that for elements $a, b \in A$ the *closed segment* $[a, b]$ is defined as

$$[a, b] = \{c \in A \mid a \leq c \leq b\}.$$

Let X be an A -metric space. A *closed segment* in X is an isometry $\gamma : [0, a] \rightarrow X$. If for every points $x, y \in X$ there exists a unique segment γ such that $\gamma(0)$ and $\gamma(a)$ then X is said to be *geodesically convex*. The image of such γ is denoted $[x, y]$.

The following definition is due to Morgan and Shalen [15].

Definition 1. *An A -tree is a geodesically convex A -metric space (X, d) such that for all $x, y, z \in X$*

- $[x, y] \cap [y, z] = [y, w]$ for some $w \in X$,
- $[x, y] \cap [y, z] = \{y\} \Rightarrow [x, z] = [x, y] \cup [y, z]$.

In the case $A = \mathbb{Z}$ this definition coincides with the definition of a simplicial tree with the path metric. Note that A viewed as an A -metric space is an A -tree.

Definition 2. *An isometric action of a group on an A -tree X is free if there are no inversions and the stabilizer of each point of X is trivial. We say that a group G is A -free if G admits such an action on an A -tree.*

The following theorem due to Chiswell provides a link between Lyndon length function and actions on A -trees.

Theorem 1. [2] *A group G has a free Lyndon length function with values in A if and only if G acts freely on an A -tree.*

2.2. Regular length functions. In this section we define regular length functions, and show some examples of groups with regular length functions. The Theorem 6 gives a host of new examples of such groups.

A length function $l : G \rightarrow A$ is called *regular* if it satisfies the *regularity* axiom:

$$(L6) \quad \forall g, f \in G, \exists u, g_1, f_1 \in G : \\ g = u \circ g_1 \text{ \& } f = u \circ f_1 \text{ \& } l(u) = c(g, f).$$

Here are several examples of groups with regular free length functions.

Example 1. *Let $F = F(X)$ be a free group on X . The length function*

$$| \cdot | : F \rightarrow \mathbb{Z},$$

where $|f|$ is a natural length of $f \in F$ as a finite word, is regular since the common initial subword of any two elements of F always exists and belongs to F .

Example 2. In [16] it was proved that the Lyndon's free $\mathbb{Z}[t]$ -group has a regular free length function with values in $\mathbb{Z}[t]$.

Example 3. [10] Let $F = F(X)$ be a free group on X . Consider an HNN-extension

$$G = \langle F, s \mid u^s = v \rangle,$$

where $u, v \in F$ are such that $|u| = |v|$ and u is not conjugate to v^{-1} . Then there is a regular free length function $l : G \rightarrow \mathbb{Z}^2$ which extends the natural integer-valued length function on F .

For more involved examples, we refer the reader to [10].

2.3. Infinite words and length functions. In this subsection at first we recall some notions from the theory of ordered abelian groups (for all the details we refer the reader to the books [5] and [12]) and then following [16] describe the construction of infinite words.

An ordered abelian group A is called *discretely ordered* if A has a minimal positive element (we denote it by 1_A). In this event, for any $a \in A$ the following hold:

- (1) $a + 1_A = \min\{b \mid b > a\}$,
- (2) $a - 1_A = \max\{b \mid b < a\}$.

Observe that if A is any ordered abelian group then $\mathbb{Z} \oplus A$ is discretely ordered with respect to the right lexicographic order.

Let A be a discretely ordered abelian group and let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^\pm = X \cup X^{-1}$. An A -word is a function of the type

$$w : [1_A, \alpha_w] \rightarrow X^\pm,$$

where $\alpha_w \in A$, $\alpha_w \geq 0$. The element α_w is called the *length* $|w|$ of w .

In the case $A = \mathbb{Z}^n$, represent \mathbb{Z}^n as a union of the chain of subgroups

$$\{0\} = A_0 < A_1 < A_2 < \cdots < A_n = \mathbb{Z}^n,$$

where

$$A_i = \{(a_1, a_2, \dots, a_n) \mid a_{i+1} = a_{i+2} = \cdots = a_n = 0\} \simeq \mathbb{Z}^i.$$

Then we say that a word w has *height* i if $|w| \in A_i - A_{i-1}$, denoted $ht(w) = i$. Note that in the case of an arbitrary discretely ordered A , an analogous definition can be given in terms of convex subgroups of A . Height of an element of a group with a free \mathbb{Z}^n -valued Lyndon length function is defined in a similar fashion.

By $W(A, X)$ we denote the set of all A -words. Observe, that $W(A, X)$ contains an empty A -word which we denote by ε .

Concatenation uv of two A -words $u, v \in W(A, X)$ is an A -word of length $|u| + |v|$ and such that:

$$(uv)(a) = \begin{cases} u(a) & \text{if } 1_A \leq a \leq |u| \\ v(a - |u|) & \text{if } |u| < a \leq |u| + |v| \end{cases}$$

An A -word w is *reduced* if $w(\beta + 1_A) \neq w(\beta)^{-1}$ for each $1_A \leq \beta < |w|$. If the concatenation uv of two reduced A -words u and v is also reduced then we write $uv = u \circ v$. We denote by $R(A, X)$ the set of all reduced A -words. Clearly, $\varepsilon \in R(A, X)$.

For $u \in W(A, X)$ and $\beta \in [1_A, \alpha_u]$ by u_β we denote the restriction of u on $[1_A, \beta]$. If $u \in R(A, X)$ and $\beta \in [1_A, \alpha_u]$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined \tilde{u}_β .

An element $\text{com}(u, v) \in R(A, X)$ is called the (*longest*) *common initial segment* of A -words u and v if

$$u = \text{com}(u, v) \circ \tilde{u}, \quad v = \text{com}(u, v) \circ \tilde{v}$$

for some (uniquely defined) A -words \tilde{u}, \tilde{v} such that $\tilde{u}(1_A) \neq \tilde{v}(1_A)$.

Now, we can define the product of two A -words. Let $u, v \in R(A, X)$. If $\text{com}(u^{-1}, v)$ is defined then

$$u^{-1} = \text{com}(u^{-1}, v) \circ \tilde{u}, \quad v = \text{com}(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined \tilde{u} and \tilde{v} . In this event put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(A, X)$.

An element $v \in R(A, X)$ is termed *cyclically reduced* if $v(1_A)^{-1} \neq v(|v|)$. We say that an element $v \in R(A, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(A, X)$ and u is cyclically reduced. Observe that a cyclic decomposition is unique (whenever it exists). We denote by $CR(A, X)$ the set of all cyclically reduced words in $R(A, X)$ and by $CDR(A, X)$ the set of all words from $R(A, X)$ which admit a cyclic decomposition.

Below we refer to A -words as *infinite words* usually omitting A whenever it does not produce any ambiguity.

The following result establishes the connection between infinite words and length functions.

Theorem 2. [16] *Let A be a discretely ordered abelian group and X be a set. Then any subgroup G of $CDR(A, X)$ has a free Lyndon length function with values in A – the restriction $L|_G$ on G of the standard length function L on $CDR(A, X)$.*

The converse of the theorem above was obtained by I.Chiswell [3].

Theorem 3. [3] *Let G have a free Lyndon length function $L : G \rightarrow A$, where A is a discretely ordered abelian group. Then there exists a length preserving embedding $\phi : G \rightarrow CDR(A, X)$, that is, $|\phi(g)| = L(g)$ for any $g \in G$.*

Corollary 1. [3] *Let G have a free Lyndon length function $L : G \rightarrow A$, where A is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \rightarrow CDR(A', X)$, where $A' = \mathbb{Z} \oplus A$ is discretely ordered with respect to the right lexicographic order and X is some set, such that, $|\phi(g)| = (0, L(g))$ for any $g \in G$.*

Theorems 2 and 3, and Corollary 1 show that a group has a free Lyndon length function if and only if it embeds into a set of infinite words and this embedding preserves the length. Moreover, it is not hard to show that this embedding also preserves regularity of the length function.

Theorem 4. [11] *Let G have a free regular Lyndon length function $L : G \rightarrow A$, where A is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \rightarrow R(A', X)$, where A' is a discretely ordered abelian group and X is some set, such that, the Lyndon length function on $\phi(G)$ induced from $R(A', X)$ is regular.*

2.4. Groups with free length functions in \mathbb{Z}^n . Let G be a finitely generated group acting freely on a \mathbb{Z}^n -tree for some $n \in \mathbb{N}$. Hence, by Corollary 1 [3], G embeds into $CDR(X, \mathbb{Z}^n)$ for some alphabet X . Then according to Corollary 4 [11], there exists a finite alphabet Y and an embedding $\phi : G \rightarrow G'$, where G' is a subgroup of $CDR(\mathbb{Z}^n, Y)$ with a regular length function, such that $|g|_G = |\phi(g)|_{G'}$ for every $g \in G$. Since G can be viewed as a subgroup of G' then in order to solve algorithmic problems for subgroups of G it is enough to solve them for subgroups of G' . So, we can assume $G = G'$ and fix it for the rest of the text.

On the other hand, the structure of groups with regular free length functions in \mathbb{Z}^n is known (see [10] for all the details).

Theorem 5 (Theorem 7 [10]). *Let G be a finitely generated group with a regular free Lyndon length function in \mathbb{Z}^n . Then G can be represented as a union of a finite series of groups*

$$G_{(1)} < G_{(2)} < \cdots < G_{(n)} = G,$$

where $G_{(1)}$ is a free group of finite rank, $G_{(i)}$ has a regular free Lyndon length function in \mathbb{Z}^i , and

$$(1) \quad G_{(i+1)} = \langle G_{(i)}, s_{i,1}, \dots, s_{i,k_i} \mid s_{i,j}^{-1} C_{i,j} s_{i,j} = \phi_{i,j}(C_{i,j}) \rangle,$$

where for each $j \in [1, k_i]$, $C_{i,j}$ and $\phi_{i,j}(C_{i,j})$ are cyclically reduced centralizers of $G_{(i)}$, $\phi_{i,j}$ is an isomorphism, and the following conditions are satisfied:

- (1) $C_{i,j} = \langle c_1^{(i,j)}, \dots, c_{m_{i,j}}^{(i,j)} \rangle$, $\phi_{i,j}(C_{i,j}) = \langle d_1^{(i,j)}, \dots, d_{m_{i,j}}^{(i,j)} \rangle$, where $\phi_{i,j}(c_k^{(i,j)}) = d_k^{(i,j)}$, $k \in [1, m_{i,j}]$ and
 $ht(c_k^{(i,j)}) = ht(d_k^{(i,j)}) < ht(d_{k+1}^{(i,j)}) = ht(c_{k+1}^{(i,j)})$, $k \in [1, m_{i,j} - 1]$,
- (2) $|\phi_{i,j}(w)| = |w|$ for any $w \in C_{i,j}$,
- (3) w is not conjugate to $\phi_{i,j}(w)^{-1}$ in $G_{(i)}$ for any $w \in C_{i,j}$,
- (4) if $A, B \in \{C_{i,1}, \phi_{i,1}(C_{i,1}), \dots, C_{i,k_i}, \phi_{i,k_i}(C_{i,k_i})\}$ then either $A = B$, or A and B are not conjugate in $G_{(i)}$,
- (5) for any $j_0 \in [1, k_i]$, the centralizer C_{i,j_0} can appear in the list
 $\{C_{i,j} \mid j \in [1, k_i]\} \cup \{\phi_{i,j}(C_{i,j}) \mid j \in [1, k_i]\}$

not more than twice.

Remark 1. In addition, from the proof of Theorem 7 [10] it follows that

- (6) $ht(s_{i,j}) = ht(s_{i,k}) = i + 1$ for any $j, k \in [1, k_i]$,
- (7) if $C_{i,j} = C_{i,k}$ then either $c(s_{i,j}, s_{i,k}) = 0$, or $s_{i,j} = s_{i,k}$.

The converse to the theorem above also holds.

Theorem 6. [10] Let H be a group with a regular free length function in $\mathbb{Z}[t]$. Let A and B be centralizers in H whose elements are cyclically reduced and such that there exists an isomorphism $\phi : A \rightarrow B$ with the following properties

- (1) a is not conjugate to $\phi_i(a)^{-1}$ in H for any $a \in A$,
- (2) $|\phi(a)| = |a|$ for any $a \in A$.

Then the group

$$(2) \quad G = \langle H, z \mid z^{-1}Az = B \rangle,$$

has a regular free length function in $\mathbb{Z}[t]$ which extends the length function on H .

Hence, we get a lot of information using the fact that every \mathbb{Z}^n -free group nicely embeds into a \mathbb{Z}^n -free group with the regular underlying length function and we are going to use this fact a lot.

Below we list other notable properties of \mathbb{Z}^n -free groups. Items 1, 2, 4, 5 can be found in Martino and Rourke's survey [14].

- (1) \mathbb{Z}^n -free groups are commutation transitive, and any abelian subgroup of a \mathbb{Z}^n -free group is free abelian of rank at most n . It follows by simple inspection of action of commuting elements.

This property holds for all A -free groups for an arbitrary abelian group A , as observed by Bass in [1].

- (2) \mathbb{Z}^n -free groups are coherent (that is, finitely generated \mathbb{Z}^n -free groups are finitely presented). While this was, likely, first observed in [14], it follows from the results of [8].
- (3) \mathbb{Z}^n -free groups are hyperbolic relative to maximal abelian subgroups. Indeed, the base change functor (see Theorem 2.3 in [14]) allows to obtain a free \mathbb{R}^n -action and then the statement follows by the result of Guirardel [6].
- (4) A finitely generated \mathbb{Z}^n -free group all of whose maximal abelian subgroups are cyclic is word hyperbolic (as are all its finitely generated subgroups).
- (5) The Word Problem is decidable in any \mathbb{Z}^n -free group.
- (6) The class of \mathbb{Z}^n -free groups is closed under amalgamated free products along maximal cyclic subgroups (n is not preserved) [13].

As we are concerned with solving the Membership Problem for \mathbb{Z}^n -free groups, Item 2 is of particular interest, since one can try to employ Kapovich–Miasnikov–Weidmann graphs of groups technique to solve the membership problem. Unfortunately, in the case of \mathbb{Z}^n -free groups this technique does not provide an effective solution because the algorithm presented in [8] requires solving the Subgroup Intersection Problem in vertex groups.

2.5. Groups with free regular length functions in \mathbb{Z}^n , a refinement. In the next subsections we introduce normal forms for elements of G . But in order to do that we need a slight refinement of Theorem 5 (we keep the notation introduced in Subsection 2.4).

First of all, for a subgroup H of G let the *height* of H be defined as

$$ht(H) = \max\{ht(h) \mid h \in H\}.$$

Obviously $ht(H)$ exists and does not exceed n . Next, denote

$$T^{(i)} = \{s_{i,j} \mid j \in [1, k_i]\}, \quad T = \bigcup_{i=1}^{n-1} T^{(i)},$$

$$\mathcal{C}^{(i)} = \{C_{i,j} \mid j \in [1, k_i]\} \cup \{D_{i,j} \mid j \in [1, k_i]\},$$

$$\mathcal{C} = \bigcup_{i=1}^{n-1} \mathcal{C}^{(i)}.$$

Consider $C \in \mathcal{C}$. We fix the generating set for C which appears in Theorem 5 and consider $c \in C$, the generator of minimal height. Observe

that every $\varepsilon \neq u \in C$ has either c or c^{-1} as an initial and terminal subword. Hence, for a fixed c we can introduce an *orientation* on C with respect to c as follows. We can represent $C - \{\varepsilon\}$ as a union $C^+ \cup C^-$, where every $u \in C^+$ has c as an initial and terminal subword, and every $u \in C^-$ has c^{-1} as an initial and terminal subword. Observe that $(C^+)^{-1} = C^-$. Obviously, there are only two possible orientations for C , the one with respect to c , and another one with respect to c^{-1} .

Now take another $D \in \mathcal{C}$ such that $w^{-1} * C * w = D$ for some $w \in G$. If C is oriented already then we can choose an orientation for D so that $w^{-1} * C^+ * w \leq D^+$. In this case we say that C and D have the same orientation.

Since $g^{-1} * f * g \neq f^{-1}$ for any $\varepsilon \neq f, g \in G$ it follows that we can orient all elements of \mathcal{C} so that any two conjugate centralizers have the same orientation. Let us fix such an orientation on \mathcal{C} .

Following [10], for $C \in \mathcal{C}$ we call $t \in T^{\pm 1}$ *attached to C* if $ht(t) > ht(C)$ and $ht(t^{-1} * C * t) = ht(C)$. If t is attached to C then we call t *left* if $c^{-1} * t = c^{-1} \circ t$, and *right* if $c * t = c \circ t$ for any $c \in C^+$.

Now, assume that $C \in \mathcal{C}^{(k)}$ and let $c \in C^+$ be the generator of maximal height. Observe that if there exists a right(left)-attached to C with respect to c element then by Lemma 15 [10] there exists $D \in \bigcup_{i=1}^k \mathcal{C}^{(i)}$ and $d \in D^+$ which is the generator of maximal height, such that c is conjugate to d in $G_{(k)}$ and D does not have right(left)-attached with respect to d elements. Next, by Lemma 16 [10] it follows that there is no element $g \in G_{(k)}$ which has any positive power of d as an initial(terminal) subword.

Starting from the upper level $n - 1$ and using the above argument, in every pair of centralizers $C, D \in \mathcal{C}^{(k)}$ such that $t^{-1} C t = D$ for some $t \in T^{(k)}$, $k \in [1, n - 1]$ we can substitute C by $w_C^{-1} C' w_C$ and D by $w_D^{-1} D' w_D$, where w_C, w_D are products of some elements from $\bigcup_{i=1}^k T^{(i)}$ and $C', D' \in \bigcup_{i=1}^k \mathcal{C}^{(i)}$ are such that any element $g \in G_{(k)}$ has neither any positive power of c as an initial subword, nor any positive power of d as a terminal subword for the generators of maximal height $c \in C'^+$, $d \in D'^+$. In other words, without loss of generality we can assume that $C' = C$ and $D' = D$ from the beginning. Finally, observe that by Theorem 5, t (note that $t \notin G_{(k)}$) is the only element of $T^{(k)}$ right-attached to C with respect to c and left-attached to D with respect to d .

Now, we split $\mathcal{C}^{(i)}$ into conjugacy classes $\mathcal{C}_j^{(i)}$, $j \in [1, K_i(G)]$ in $G_{(i+1)}$. Observe that it is possible that $A \in \mathcal{C}_j^{(i)}$ is a subgroup of $A' \in \mathcal{C}_{j'}^{(i')}$ for some $i \leq i'$. Also, by the above assumption, all elements of $\mathcal{C}_j^{(i)}$ for

$i \in [1, n-1], j \in [1, K_i(G)]$ have the same orientation. Moreover, any two $A, B \in \mathcal{C}_j^{(i)}$ have the same height, and $w^{-1} * A * w = B$, $w \neq \varepsilon$ implies $ht(w) > ht(A)$. Denote by $T_j^{(i)}$ the corresponding subset of $T^{(i)}$.

Denote $ht(\mathcal{C}_j^{(i)}) = ht(A)$, where $A \in \mathcal{C}_j^{(i)}$, $j \in [1, K_i(G)]$. Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{K(G)}$ be a reordering of the set of conjugacy classes

$$\{\mathcal{C}_1^{(1)}, \dots, \mathcal{C}_{K_1(G)}^{(1)}, \mathcal{C}_1^{(2)}, \dots, \mathcal{C}_{K_2(G)}^{(2)}, \dots, \mathcal{C}_1^{(n-1)}, \dots, \mathcal{C}_{K_{n-1}(G)}^{(n-1)}\}$$

such that $\mathcal{C}_j^{(i)}$ has lower number than $\mathcal{C}_{j'}^{(i')}$ whenever $i < i'$, while the order relation on the conjugacy classes within the same $\mathcal{C}^{(i)}$ is arbitrary. Assume that the corresponding sets $T_j^{(i)}$ are reordered accordingly. Observe that

$$\mathcal{C} = \bigcup_{i=1}^{K(G)} \mathcal{C}_i, \quad T = \bigcup_{i=1}^{K(G)} T_i.$$

Now, using the notation just introduced we are ready to reformulate Theorem 5 pointing out the properties of the new presentation of G necessary in the further considerations.

Theorem 7. *Let T_i and \mathcal{C}_i , where $i \in [1, K(G)]$, be defined as above. Then G can be represented as a union of a finite series of groups*

$$(3) \quad G_1 < G_2 < \dots < G_{K(G)} = G,$$

where G_1 is a free group of finite rank, and

$$G_{i+1} = \langle G_i, T_i \mid t^{-1} C_t t \stackrel{\phi_t}{=} D_t, \quad t \in T_i, \quad C_t, D_t \in \mathcal{C}_i \rangle,$$

where $\phi_t : C_t \rightarrow D_t$ is an isomorphism and the following conditions are satisfied:

- (1) each G_i is a group with a free regular length function,
- (2) C_t and D_t are cyclically reduced centralizers for any $t \in T_i$,
- (3) $C_t = \langle c_1^{(t)}, \dots, c_{k_t}^{(t)} \rangle$, $D_t = \langle d_1^{(t)}, \dots, d_{k_t}^{(t)} \rangle$, where all $c_i^{(t)}, d_i^{(t)}$ are cyclically reduced,

$$ht(c_i^{(t)}) = ht(d_i^{(t)}) < ht(d_{i+1}^{(t)}) = ht(c_{i+1}^{(t)}), \quad i \in [1, k_t - 1],$$

$$\text{and } \phi(c_i^{(t)}) = d_i^{(t)},$$

- (4) $|\phi_t(c)| = |c|$ for any $c \in C_t$,
- (5) for any $s \in T_i$, the centralizer C_s can appear in the list

$$\{C_t \mid t \in T_i\} \cup \{D_t \mid t \in T_i\}$$

not more than twice,

- (6) $ht(s) = ht(t)$ for any $s, t \in T_i$, and if $C_s = C_t$ then either $c(s, t) = 0$, or $s = t$,

- (7) any element $g \in G_i$ may have only a bounded positive power of $c_{k_t}^{(t)}$ as an initial subword and only a bounded positive power of $d_{k_t}^{(t)}$ as a terminal subword.

Proof. (1) – (5) follow from Theorem 5, (6) follows from Remark 1, and (7) follows from the discussion above. \square

2.6. Normal forms in G_{i+1} . Fix $i \in [1, K(G)]$. From now on we are going to use the following notation. Let $ST_i \subset \langle T_i \rangle$ be the set of elements $w \in \langle T_i \rangle$ such that there exist $C, D \in \mathcal{C}_i$ with the property $w^{-1} * C * w = D \in \mathcal{C}_i$. Obviously, $\varepsilon \in ST_i$ and $(ST_i)^{-1} = ST_i$. For every $w \in ST_i$, if $w^{-1} * C * w = D$ for $C, D \in \mathcal{C}_i$ then denote $C_w = C$, $D_w = D$, and denote by u_w , v_w the generators of C_w and D_w of maximal height such that $w^{-1} * u_w * w = v_w$, $u_w * w = u_w \circ w$, $w * v_w = w \circ v_w$. It is easy to see that $C_{w^{-1}} = D_w$, $D_{w^{-1}} = C_w$ and $u_{w^{-1}} = v_w^{-1}$, $v_{w^{-1}} = u_w^{-1}$. Also, let

$$U_i = \{u_t, v_t \mid t \in T_i\}.$$

According to Theorem 7, G_{i+1} is obtained from G_i as a multiple HNN-extension, so every $g \in G_{i+1} - G_i$ has the following representation

$$(4) \quad g = g_1 * w_1 * g_2 * \cdots * w_k * g_{k+1},$$

where $g_j \in G_i$, $j \in [1, k+1]$, $w_j \in ST_i$, $j \in [1, k]$, and if $D_{w_j} = C_{w_{j+1}}$ then $g_j \notin D_{w_j}$ for any $j \in [1, k-1]$.

Lemma 8. *Let $g \in G_i$ for some $i \in [1, K(G)]$. If $c \in C$, $d \in D$ are generators of maximal height for arbitrary $C, D \in \mathcal{C}_i$ then there exists $N(g) > 0$ such that for every $r \geq N(g)$*

- (a) $c^r * g = c^{r-N(g)} \circ (c^{N(g)} * g)$,
- (b) $c^r * g * d^r = c^{r-N(g)} \circ (c^{N(g)} * g * d^{N(g)}) \circ d^{r-N(g)}$, provided $g \notin C$ if $c = d^{-1}$.

Proof. Follows from Theorem 7 (7). \square

By Lemma 8 there exist $N_j, M_j \in \mathbb{Z}$, $j \in [1, k]$ such that $M_j, N_j \geq N(g_j)$ and (4) can be transformed into

$$(5) \quad g = (g_1 * u_{w_1}^{N_1}) \circ (u_{w_1}^{-N_1} * w_1 * v_{w_1}^{-M_1}) \circ (v_{w_1}^{M_1} * g_2 * u_{w_2}^{N_2}) \circ \cdots \circ (u_{w_k}^{-N_k} * w_k * v_{w_k}^{-M_k}) \circ (v_{w_k}^{M_k} * g_{k+1}).$$

Since N_j, M_j only satisfy the condition $M_j, N_j > N(g_j)$, and g_j may have elements of $D_{w_{j-1}}$ and C_{w_j} respectively as initial and terminal subwords it follows that (5) is not a unique representation of g .

Our goal is to obtain a unique representation for g .

Obviously $u^N * w * v^M = w * v^{M+N}$, $(u^N * w * v^M)^{-1} = w^{-1} * u^{-(M+N)}$ for any $w \in ST_i$, $N, M \in \mathbb{Z}$, and $u = u_w^\delta$, $v = v_w^\delta$, $\delta \in \{-1, 1\}$, so, denote

$$w(r) = w * v^r.$$

Notice that $w(r) \in G_{i+1}$ for every $r \in \mathbb{Z}$, and $w(r)$ has any natural powers of u_w and v_w respectively as an initial and terminal subword. So, we can use the notation $C_{w(r)} = C_w$, $D_{w(r)} = D_w$, $u_{w(r)} = u_w$, $v_{w(r)} = v_w$.

Now, let

$$(6) \quad g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1},$$

where $g_j \in G_i$, $j \in [1, k+1]$, $w_j \in ST_i$, $j \in [1, k]$ and if $D_{w_j} = C_{w_{j+1}}$ then $g_j \notin D_{w_j}$ for any $j \in [1, k-1]$. Then, (6) can be characterized by a tuple

$$\{|w_1(r_1)|, \dots, |w_k(r_k)|\},$$

where $|w_j(r_j)| \in \mathbb{Z}^n$, $j \in [1, k]$. It is easy to see that any other representation of g as a reduced infinite word is very similar to (6). In particular, the following statement holds.

Lemma 9. *Let $g \in G_{i+1} - G_i$ have two representations*

$$g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1},$$

where $g_j \in G_i$, $j \in [1, k+1]$, $w_j \in ST_i$, $j \in [1, k]$ and if $D_{w_j} = C_{w_{j+1}}$ then $g_j \notin D_{w_j}$ for any $j \in [1, k-1]$, and

$$g = h_1 \circ z_1(q_1) \circ h_2 \circ \cdots \circ z_m(q_m) \circ h_{m+1},$$

where $h_j \in G_i$, $j \in [1, m+1]$, $z_j \in ST_i$, $j \in [1, m]$ and if $D_{z_j} = C_{z_{j+1}}$ then $h_j \notin D_{z_j}$ for any $j \in [1, m-1]$. Then $k = m$ and $w_j = z_j$, $j \in [1, k]$. Moreover, $h_j = v_j * g_j * u_j$, where $v_j \in D_{w_{j-1}}$, $u_j \in C_{w_j}$.

Proof. We prove by induction on $k + m$. Observe that if $k + m = 0$ then there is nothing to prove, so, let $k + m > 0$. We have

$$(g_1 \circ w_1(r_1) \circ \cdots \circ w_k(r_k) \circ g_{k+1}) * (h_1 \circ z_1(q_1) \circ \cdots \circ z_m(q_m) \circ h_{m+1})^{-1} = \varepsilon,$$

that is,

$$(g_1 \circ w_1(r_1) \circ \cdots \circ w_k(r_k) \circ g_{k+1}) * (h_{m+1}^{-1} \circ z_m^{-1}(-q_m) \circ \cdots \circ z_1^{-1}(-q_1) \circ h_1^{-1}) = \varepsilon,$$

which is only possible if $D_{w_k}^+ = D_{z_m}^+$ provided $d = g_{k+1} * h_{m+1}^{-1} \in D_{w_k}$. By definition of ST_i and by Theorem 7 (6) we have $w_k = z_m$. Let $c = w_k(r_k) * (g_{k+1} * h_{m+1}^{-1}) * z_m^{-1}(-q_m) \in C_{w_k}$, so we have

$$(g_1 \circ w_1(r_1) \circ \cdots \circ w_{k-1}(r_{k-1}) \circ g_k) * c * (h_m^{-1} \circ z_{m-1}^{-1}(-q_{m-1}) \circ \cdots \circ z_1^{-1}(-q_1) \circ h_1^{-1}) = \varepsilon.$$

Now, we have two representations of $h \in G_{i+1} - G_i$, where

$$h = g_1 \circ w_1(r_1) \circ \cdots \circ w_{k-1}(r'_{k-1}) \circ g'_k,$$

$$h = h_1 \circ z_1(q_1) \circ \cdots \circ z_{m-1}(q_{m-1}) \circ h_m,$$

where $w_{k-1}(r'_{k-1}) \circ g'_k = (w_{k-1}(r_{k-1}) \circ g_k) * d$, and by the induction hypothesis we get $m - 1 = k - 1$, $w_j = z_j$, $j \in [1, k - 1]$. So, the statement follows. \square

From Lemma 8 it follows that any representation of $g \in G_{i+1} - G_i$ can be associated with a particular k -tuple for a fixed $k \in \mathbb{N}$. Now, representation (6) is called the *normal form* if the k -tuple

$$\{|w_1(r_1)|, \dots, |w_k(r_k)|\}$$

is maximal in the left lexicographic order among k -tuples corresponding to all representations of g . Existence of the normal representation (form) follows from Lemma 16 [10]. The (uniquely defined) number k is called the *level $i + 1$ syllable length* of g . Elements of G_i are said to have level $i + 1$ syllable length 0. Normal forms satisfies the following properties.

Remark 2. *Normal forms are not unique since, while w_j are defined uniquely by Lemma 9, elements g_j are only defined up to multiplication $v * g_j * u$, where $v \in D(w_{j-1})$ and $u \in C(w_j)$.*

Lemma 10. *If*

$$g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

is a normal form for g then

- (a) g_1 does not have $u_{w_1}^{\pm 1}$ as a terminal subword,
- (b) $w_{i-1} \circ g_1$ does not have $u_{w_i}^{\pm 1}$ as a terminal subword for $i \in [2, k]$,
- (c) $g_{i+1} \circ w_{i+1}(r_{i+1})$ does not have $u_{w_{i+1}}^{\pm 1}$ as an initial subword for $i \in [1, k - 1]$,
- (d) g_{k+1} does not have $v_{w_k}^{\pm 1}$ as an initial subword.

Proof. Since there is no cancellation in the products $g_i \circ w_i(r_i)$, $w_i(r_i) \circ g_{i+1}$, $i \in [1, k]$, if at least one of the properties listed above does not hold then at least one of $|w_i(r_i)|$, $i \in [1, k]$ can be made bigger by a finite sum of $|u|$ (even at the expense of $|w_{i+1}(r_{i+1})|$), where u can be taken to be a generator of any $C \in \mathcal{C}_i$ of maximal height. This implies that $\{|w_1(r_1)|, \dots, |w_k(r_k)|\}$ is not maximal in the left lexicographic order - a contradiction. \square

Suppose

$$(7) \quad g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

is a normal form for g , where g is cyclically reduced. Then, obviously

$$(8) \quad \begin{aligned} g^2 = & g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ (g_{k+1} \circ g_1) \\ & \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1} \end{aligned}$$

is a representation of g^2 which may not be a normal form. So, we call (7) *cyclically normal* if (8) is normal.

Lemma 11. *Let $u \in C^+$ and $v \in D^+$ be generators of $C, D \in \mathcal{C}_i$, $C \neq D$ of maximal height such that $t^{-1} * u * t = v$ for some $t \in \langle T_i \rangle$. Let $w \in G_i$ be such that $|w| < |u| = |v|$.*

- (a) *If $u^\alpha * w * u^\beta = u^\alpha \circ w \circ u^\beta$, $\alpha, \beta \in \mathbb{Z}$ and $w \circ u^\beta$ has $u^{\pm 1}$ as an initial subword then $u^\alpha \circ w$ cannot have $u^{\pm 1}$ as a terminal subword provided $w \notin C$.*
- (b) *If $u^\alpha * w * v^\beta = u^\alpha \circ w \circ v^\beta$, $\alpha, \beta \in \mathbb{Z}$ and $w \circ v^\beta$ has $u^{\pm 1}$ as an initial subword then $u^\alpha \circ w$ cannot have $v^{\pm 1}$ as a terminal subword.*

Proof. (a) Suppose $\beta > 0$.

If $w \circ u^\beta$ has u as an initial segment then $\alpha > 0$ and $u = w \circ u_1$, $u = u_1 \circ u_2$. Observe that $u^\alpha \circ w$ cannot end with u^{-1} but if it does end with u then $u = u_3 \circ w$, $u = u_4 \circ u_3$. Combining all four representations of u we get $w = u_2$ which implies $u = w \circ u_1 = u_1 \circ w$, that is, $w \in C$ - a contradiction.

If $w \circ u^\beta$ has u^{-1} as an initial segment then $\alpha < 0$ and $u^{-1} = w \circ u_1$, $u = u_1 \circ u_2$ which implies $u_1 = \varepsilon$ and $w = u_2^{-1} \in C$ - a contradiction.

Suppose $\beta < 0$.

If $w \circ u^\beta$ has u as an initial segment then $\alpha > 0$ and $u = w \circ u_1$, $u^{-1} = u_1 \circ u_2$ which implies $u_1 = \varepsilon$ and $w = u_2^{-1} \in C$ - a contradiction.

If $w \circ u^\beta$ has u^{-1} as an initial segment then $\alpha < 0$ and $u^{-1} = w \circ u_1$, $u^{-1} = u_1 \circ u_2$. Observe that $u^\alpha \circ w$ cannot end with u but if it does end with u^{-1} then $u^{-1} = u_3 \circ w$, $u^{-1} = u_4 \circ u_3$. Combining all four representations of u^{-1} we get $w = u_2$ which implies $u^{-1} = w \circ u_1 = u_1 \circ w$, that is, $w \in C$ - a contradiction.

(b) Suppose $\alpha > 0$, $\beta > 0$. If $w \circ v^\beta$ has u as an initial segment then $u = w \circ v_1$, $v = v_1 \circ v_2$, $|w| = |v_2|$. Observe that $u^\alpha \circ w$ cannot end with v^{-1} but if it does end with v then $v = u_2 \circ w$, $u = u_1 \circ u_2$, $|w| = |u_1|$. Now it follows that $w = u_1$, $v_1 = u_2$ and $v = u_2 \circ u_1$ which is a contradiction since C and D are not conjugate in G_i .

Suppose $\alpha > 0$, $\beta < 0$. If $w \circ v^\beta$ has u as an initial segment then $u = w \circ v_1$, $v^{-1} = v_1 \circ v_2$, $|w| = |v_2|$. Observe that $u^\alpha \circ w$ cannot end with v but if it does end with v^{-1} then $v^{-1} = u_2 \circ w$, $u = u_1 \circ u_2$, $|w| = |u_1|$. Now it follows that $w = v_2$, $v_1 = u_2$ and $u = v_2 \circ v_1$, that is, u is a conjugate of v^{-1} . Finally, since u is a conjugate of v it follows that v is a conjugate of v^{-1} - a contradiction.

The cases $\alpha < 0$, $\beta > 0$ and $\alpha < 0$, $\beta < 0$ can be considered in a similar way. □

Lemma 12. *Let*

$$g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

*be a normal form of a cyclically reduced $g \in G_{i+1} - G_i$. Then there exists a cyclic permutation of g whose normal form is cyclically normal unless $k = 1$ and $g_2 * g_1 \in D_{w_1} = C_{w_1}$.*

Proof. Without loss of generality we can assume $g_1 = \varepsilon$ and that g_{k+1} does not have u_{w_1} as a terminal subword (using cyclic permutation we can always obtain these properties).

If $v_{w_k} = u_{w_1}$ and $g_{k+1} \in D_{w_k}$ then applying a cyclic permutation we reduce the number of entries of elements from $\langle T_i \rangle$ provided $k > 1$. Otherwise, $g = w_1(r_1) \circ g_2$, $g_2 = g_{k+1} \in D_{w_1}$.

So we can assume that either $v_{w_k} \neq u_{w_1}$ or $g_{k+1} \notin D_{w_k}$

$$(1) |g_{k+1}| \geq |u_{w_1}|$$

Since $w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$ is normal it follows that g_{k+1} does not have u_{w_k} as an initial and u_{w_1} as a terminal subword. Hence,

$$w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1} \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

is normal.

$$(2) |g_{k+1}| < |u_{w_1}|$$

If $g_{k+1} \circ w_1(r_1)$ does not have v_{w_k} as an initial segment then

$$w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

is cyclically normal.

Suppose $g_{k+1} \circ w_1(r_1)$ has v_{w_k} as an initial segment. We take a cyclic permutation

$$g' = g_2 \circ w_2(r_2) \circ \cdots \circ w_k(r_k) \circ g_{k+1} \circ w_1(r_1)$$

of g so that

$$g_2 \circ w_2(r_2) \circ \cdots \circ w_k(r_k + 1) \circ b \circ w_1(r_1 - 1)$$

is a normal form for g' , where $v_{w_k} = g_{k+1} \circ a$, $u_{w_1} = a \circ b$. But then

$$(g_2 \circ w_2(r_2) \circ \cdots \circ w_k(r_k + 1) \circ b \circ w_1(r_1 - 1)) \circ (g_2 \circ w_2(r_2) \circ \cdots \circ w_k(r_k + 1) \circ b \circ w_1(r_1 - 1))$$

is a normal form for $(g')^2$ by Lemma 11. □

2.7. Normal forms in G . Recall that G can be represented as a union of a finite series of groups (3)

$$G_1 < G_2 < \cdots < G_{K(G)} = G$$

satisfying the properties shown in Theorem 7. In particular $G_1 = \langle X \rangle$ is a free group.

Using the series (3) we can introduce a *standard decomposition* $\pi(g)$ for $g \in G$, where $\pi(g) \in (\mathcal{B}(G)^{\pm 1})^*$ and

$$\mathcal{B}(G) = X \bigcup \left(\bigcup_i T_i \right) \bigcup (\{u^n \mid n \in \mathbb{Z}, u \in U\}),$$

where

$$U = \bigcup_i U_i,$$

$$U_i = \{u \in C \mid C \in \mathcal{C}_i, u \text{ is a generator of } C \text{ of maximal height}\}.$$

If $g \in G_1$ then

$$g = x_1 \cdots x_k, \quad x_i \in X^{\pm 1},$$

which is a reduced word in a free group G_1 , so we put

$$\pi(g) = x_1 \cdots x_k.$$

Assuming inductively that $\pi(f)$ is constructed for any $f \in G_i$ we consider $g \in G_{i+1} - G_i$. By the results of the previous subsection we can compute a normal form for g

$$g = g_1 \circ w_1(r_1) \circ g_2 \circ \cdots \circ w_k(r_k) \circ g_{k+1}$$

satisfying the properties shown in Lemma 10. Since g_i , $i \in [1, k+1]$ are elements of G_i it follows that $\pi(g_i)$, $i \in [1, k+1]$ are defined and we put

$$\pi(g) = \pi(g_1) w_1(r_1) \pi(g_2) \cdots w_k(r_k) \pi(g_{k+1}).$$

Observe that in general, for a cyclically reduced $g \in G$ we have $\pi(g^2) \neq \pi(g)^2$. In particular, if $u \in G_{i+1}$ is such that there exists $t \in T_{i+1}$ such that $u = u_t$ then $\pi(u^2)$ does not have to be equal to $\pi(u)^2$. On the other hand, by Lemma 12 every $g \in G_{i+1} - G_i$ has a cyclic permutation whose normal form is cyclically normal unless

$g = w \circ h$, where $w \in ST_i$ and $h \in D_w = C_w$. Since all centralizers from \mathcal{C}_{i+1} can be taken up to conjugation (cyclic permutation) then we can always assume that $\pi(u^2) = \pi(u)^2$ unless $u = w \circ h$, where $w \in ST_i$ and $h \in D_w = C_w$. Observe that in the latter case t has any natural power of u as an initial and terminal subword and without loss of generality we can assume $u = w$ which is the generator of maximal height in $C_t = C_{G_{i+1}}$. Hence, $\pi(w^2) = \pi(w)^2$.

3. $\mathcal{B}(G)$ -GRAPHS

Let us fix a group G with a free regular length function in \mathbb{Z}^n for the rest of this section. According to the previous section, every $g \in G$ can be written as a word in the alphabet $\mathcal{B}(G)^{\pm 1}$. In this section we introduce graphs whose edges are labeled by $\mathcal{B}(G)$ and show how such graphs can be associated with subgroups of G .

3.1. Labeled graphs. Using the notation introduced in the previous section we adjust basic notions from [7] to the case of groups acting freely on \mathbb{Z}^n -trees.

Definition 3. *By a $\mathcal{B}(G)$ -labeled directed graph ($\mathcal{B}(G)$ -graph) Γ we mean the following.*

- (1) Γ is a combinatorial graph, where every edge has a direction and is labeled by a letter from $\mathcal{B}(G)$, denoted $\mu(e)$,
- (2) for each edge e of Γ we denote the origin of e by $o(e)$ and the terminus of e by $t(e)$.

For each edge e of $\mathcal{B}(G)$ -graph we introduce a formal inverse e^{-1} of e with label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e)$, $t(e^{-1}) = o(e)$, that is, the direction of e^{-1} is reversed with respect to the direction of e . For the new edge e^{-1} we set $(e^{-1})^{-1} = e$. The new graph, endowed with this additional structure we denote by $\widehat{\Gamma}$. In fact, usually we will abuse notation by disregarding the difference between Γ and $\widehat{\Gamma}$.

Now we have a partition $E(\widehat{\Gamma}) = E(\Gamma) \cup \overline{E(\Gamma)}$ and we say that edges of Γ are *positively oriented* in $\widehat{\Gamma}$, while their formal inverses e^{-1} are *negatively oriented* in $\widehat{\Gamma}$.

Definition 4. *A path p in Γ is a sequence of edges $p = e_1 \cdots e_k$, where each e_i is an edge of $\widehat{\Gamma}$ and the origin of each e_i is the terminus of e_{i-1} .*

Observe that $\mu(p) = \mu(e_1) \cdots \mu(e_k)$ is a word in the alphabet $\mathcal{B}(G)$ and we denote by $\mu(p)$ the reduced infinite word $\mu(e_1) * \cdots * \mu(e_k)$.

We will be using two different notions of the length of a path $p = e_1 \cdots e_k$ in Γ

- (1) *combinatorial length* $|p|$ set equal to k , and
- (2) *word length* $wl(p) = \sum_{i=1}^k l(\mu(e_i))$.

In fact, from these two definitions above two possible meanings of irreducible path arise: irreducible path in combinatorial sense and irreducible path in the sense that its label is viewed as an infinite word in $CDR(\mathbb{Z}^n, X)$ is reduced. Here are formal definitions.

Definition 5. A path $p = e_1 \cdots e_k$ in a $\mathcal{B}(G)$ -graph Γ is called *reduced* if $e_i \neq e_{i+1}^{-1}$ for all $i \in [1, k-1]$.

Definition 6. A path $p = e_1 \cdots e_k$ in a $\mathcal{B}(G)$ -graph Γ is called *label reduced* if

- (1) p is reduced,
- (2) if $t = \mu(e_l) = \mu(e_m)^{-1}$, $t \in T_i^{\pm 1}$, $l < m$, $l, m \in [1, k]$ and $q = e_{l+1} \cdots e_{m-1}$ does not contain edges labeled by $T_j^{\pm 1}$, $j \geq i$ then $w = \overline{\mu(q)} \notin D_t$, moreover, $t * w = t \circ w$, $w * t^{-1} = w \circ t^{-1}$.

Recall that a graph labeled by letters from X^\pm defines a language of words over X^\pm . This language can be put into correspondence with a subgroup of a free group $F(X)$. In the present subsection we generalize this concept to $\mathcal{B}(G)$ -graphs.

Definition 7. Let Γ be a $\mathcal{B}(G)$ -graph and let v, v' be vertices of Γ . We define the language of Γ with respect to v to be

$$L(\Gamma, v) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\}.$$

We also put

$$L(\Gamma, v, v') = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v'\}.$$

If w belongs to $L(\Gamma, v)$, we will also sometimes say that w is *accepted* by (Γ, v) (or just by Γ if v is fixed).

The following result establishes a connection between $\mathcal{B}(G)$ -graphs and subgroups of G .

Lemma 13. Let Γ be a finite $\mathcal{B}(G)$ -graph and let $v \in V(\Gamma)$. Then $L(\Gamma, v)$ is a subgroup of G .

Proof. Straightforward verification. □

Similarly, $L(\Gamma, v, v_1)$ is a left $L(\Gamma, v')$ -coset and a right $L(\Gamma, v)$ -coset of G .

3.2. Free foldings. Here we define free (partial) foldings and partially folded (\mathbb{Z}^n, X) -graphs. Observe that the definition of a partial folding below is exactly the same as the corresponding definition of a folding in free groups (see [7]).

Let Γ be a $\mathcal{B}(G)$ -graph. Suppose v_0 is a vertex of Γ and f_1, f_2 are two distinct edges of Γ such that $o(f_1) = o(f_2) = v_0$, $\mu(f_1) = \mu(f_2) \in \mathcal{B}^\pm$. Let h_i be the positive edge of Γ corresponding to f_i (that is, $h_i = f_i$ if f_i is positive and $h_i = f_i^{-1}$ if f_i is negative).

Let Δ be a $\mathcal{B}(G)$ -graph with the following sets of vertices and edges.
 $V(\Delta) = (V(\Gamma) - \{t(f_1), t(f_2)\}) \cup \{v\}$, $E(\Delta) = (E(\Gamma) - \{h_1, h_2\}) \cup \{h\}$.

The endpoints and arrows for the edges of Δ are defined in the following way. Let $e \in E(\Delta)$, $e \neq h$ then

- (1) we put $o_\Delta(e) = o_\Gamma(e)$ if $o_\Gamma(e) \neq t(f_i)$ and $o_\Delta(e) = v$ if $o_\Gamma(e) = t(f_i)$ for some i ,
- (2) we put $t_\Delta(e) = t_\Gamma(e)$ if $t_\Gamma(e) \neq t(f_i)$ and $t_\Delta(e) = v$ if $t_\Gamma(e) = t(f_i)$ for some i .

For the edge h we put $o_\Delta(h) = v_0$, $t_\Delta(h) = v$ if $h_1 = f_1$, $h_2 = f_2$ and $o_\Delta(h) = v$, $t_\Delta(h) = v_0$ otherwise.

We define labels on the edges of Δ as follows: $\mu_\Delta(e) = \mu_\Gamma(e)$ if $e \neq h$ and $\mu_\Delta(h) = \mu_\Gamma(h_1) = \mu_\Gamma(h_2)$.

In other words we obtain Δ by identification of two edges f_1 and f_2 in Γ . In this situation we say that Δ is obtained from Γ by a *free folding* (or by *freely folding the edges f_1 and f_2*).

There can be introduced a notion of a *morphism* between two $\mathcal{B}(G)$ -graphs. That is, if Γ_1, Γ_2 are $\mathcal{B}(G)$ -graphs then a map $\theta : \Gamma_1 \rightarrow \Gamma_2$ is called a *morphism* of $\mathcal{B}(G)$ -graphs, if θ sends vertices to vertices, directed edges to directed edges, preserves labels of directed edges, and has the property that $o(\theta(e)) = \theta(o(e))$, $t(\theta(e)) = \theta(t(e))$ for any edge e of Γ_1 .

If ϕ is a free folding defined above then it is easy to see that ϕ is a morphism between Γ and Δ .

Lemma 14. *Let Γ_1 be a $\mathcal{B}(G)$ -graph obtained by a free folding from a graph Γ . Let v, v' vertices of Γ and v_1, v'_1 be the corresponding vertices of Γ_1 . Then the following hold.*

- (1) *If Γ is connected then Γ_1 is connected.*
- (2) *Let p be the path from v to v' in Γ with label w . Then the edgewise image of p in Γ_1 is a path from v_1 to v'_1 with label w .*
- (3) *If Γ is a finite $\mathcal{B}(G)$ -graph, then the number of edges in Γ_1 is one less than the number of edges in Γ , that is, any free folding decreases the number of edges in Γ .*

Proof. Follows directly from the definition of a free folding. \square

Definition 8. $\mathcal{B}(G)$ -graph Γ is called *freely folded* if there exist no two edges e_1 and e_2 in Γ with $\mu(e_1) = \mu(e_2)$ such that $o(e_1) = o(e_2)$ or $t(e_1) = t(e_2)$.

Obviously, Γ is a freely folded $\mathcal{B}(G)$ -graph if and only if one cannot perform any free folding in Γ . Moreover the following proposition is true.

Lemma 15. Let Γ be a $\mathcal{B}(G)$ -graph, which has only a finite number of edges. Then there exists a freely folded $\mathcal{B}(G)$ -graph Δ , which can be obtained from Γ by a finite number of free foldings.

Proof. Since Γ has a finite number of edges by Lemma 14, any $\mathcal{B}(G)$ -graph Γ_1 obtained from Γ by a free folding has fewer edges. This provides one with an inductive argument based on the number of edges in Γ . \square

Lemma 16. Let Γ be a finite $\mathcal{B}(G)$ -graph and let $v, v' \in V(\Gamma)$. Let Δ be a $\mathcal{B}(G)$ -graph obtained from Γ by a single free folding so that $v_1, v'_1 \in V(\Delta)$ correspond to v, v' . Then

$$L(\Gamma, v, v') = L(\Delta, v_1, v'_1).$$

Proof. Similar to the proof of Lemma 3.4 [7]. \square

3.3. C -components. In the present subsection we fix $C \in \mathcal{C}_m$ and concentrate on subgraphs of $\mathcal{B}(G)$ -graphs which consist of edges labeled by exponents u^k , $u \in U_i \cap C$, $i \leq m$.

Let Γ be a $\mathcal{B}(G)$ -graph and let $C \in \mathcal{C}_m$. We say that $v_0, v_1 \in V(\Gamma)$ are C -equivalent (denote $v_0 \sim_C v_1$) if there exists a path p in Γ such that $o(p) = v_0$, $t(p) = v_1$, and every edge of p is labeled by a power of $u \in U_i \cap C$, $i \leq m$. In other words, if $v_0 \sim_C v_1$ then there exists a path connecting them, whose reduced label is an element of C .

One can take the subgraph of Γ spanned by vertices C -equivalent to $v \in V(\Gamma)$ and remove from it all edges labeled by anything except for $u \in U_i \cap C$, $i \leq m$. The resulting subgraph of Γ we denote by $Comp_C(v)$ and call the C -component of v .

Definition 9. Let Γ be a $\mathcal{B}(G)$ -graph and $v \in V(\Gamma)$, $v_0, v_1 \in V(Comp_C(v))$. We define a set $H_C(v_0)$ associated with v_0 as

$$H_C(v_0) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } Comp_C(v) \text{ from } v_0 \text{ to } v_0\}.$$

We also put

$$H_C(v_0, v_1) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \text{Comp}_C(v) \text{ from } v_0 \text{ to } v_1\}.$$

Observe that even when p is a reduced path in $\text{Comp}_C(v)$ its label $\overline{\mu(p)}$ may be the empty infinite word.

Lemma 17. *Let Γ be a $\mathcal{B}(G)$ -graph and $v \in V(\Gamma)$, $v_0 \in V(\text{Comp}_C(v))$. Then*

- (1) $H_C(v_0)$ is a subgroup of C ,
- (2) if $v_1 \in V(\text{Comp}_C(v))$ then $H_C(v_0) = H_C(v_1)$.

Proof. Straightforward verification. □

It follows from Lemma 17 that one can associate a subgroup of C with any finite C -component Q in a $\mathcal{B}(G)$ -graph Γ . We can denote this subgroup by $H_C(Q)$.

3.4. Reduced C -components. Suppose Q is a partially folded C -component of a $\mathcal{B}(G)$ -graph Γ and fix $v_0 \in V(Q)$. For a vertex $v \in V(Q)$ take a path p_v such that $o(p_v) = v_0$, $t(p_v) = v$. So $\overline{\mu(p_v)} \in C$ and obviously for any other path q_v such that $o(q_v) = v_0$, $t(q_v) = v$ we have $\overline{\mu(p_v)} * \overline{\mu(q_v)} \in H_C(Q)$ since $p_v q_v^{-1}$ is a loop at v_0 . Hence, every vertex $v \in V(Q)$ can be associated with a coset $H_C(v_0, v) = \overline{\mu(p_v)} * H_C(Q)$ in C by $H_C(Q)$. Denote this system of cosets by $C_Q(v_0)$.

It can happen that two distinct vertices of Q correspond to the same coset in $C_Q(v_0)$, that is,

$$\overline{\mu(p_{v_1})} * H_C(Q) = \overline{\mu(p_{v_2})} * H_C(Q), \quad v_1 \neq v_2.$$

Observe that in this case v_1 and v_2 correspond to the same coset in $C_Q(v)$ for any $v \in V(Q)$ and we can apply a *simple reduction* of Q , which is an identification of the vertices v_1 and v_2 . It is easy to see that simple reductions are morphisms of graphs.

Lemma 18. *Let Q be a C -component of a finite $\mathcal{B}(G)$ -graph Γ . Let $v_1, v_2 \in V(Q)$ correspond to the same coset in $C_Q(v_0)$ for some $v_0 \in V(Q)$, and let Δ be a $\mathcal{B}(G)$ -graph obtained from Γ by a simple reduction ϕ of Q such that $\phi(v_1) = \phi(v_2)$. If $v_0, v \in V(\Gamma)$ and $v'_0 = \phi(v_0)$, $v' = \phi(v)$ then*

$$L(\Gamma, v_0, v) = L(\Delta, v'_0, v').$$

Proof. Straightforward verification similar to Lemma 17. □

If every coset in $C_Q(v_0)$, $v_0 \in V(Q)$ corresponds to a unique vertex of Q then we say that Q is a *reduced C -component*. Since simple

reductions decrease the number of vertices it follows that after finitely many reductions we obtain a reduced C -component.

3.5. ST_m -complexes. A path p in a $\mathcal{B}(G)$ -graph Γ we call an ST_m -path if $\mu(p) \in ST_m$, that is, if $p = e_1 \cdots e_k$ then $\mu(e_i) = t_i \in T_m^{\pm 1}$ and $D_{t_i} = C_{t_{i+1}}$, $i \in [1, k-1]$. If Q is a non-trivial C -component of Γ , where $C \in \mathcal{C}_m$, then we call Q *attached to an ST_m -path p* if $o(p)$ belongs to $V(Q)$ and $C = C_w$, where $w = \mu(p) \in ST_m$. A connected subgraph P of Γ is called an ST_m -subgraph if for every two vertices $v_1, v_2 \in V(P)$ there exists a path

$$p = s_0 p_1 s_1 p_2 \cdots p_k s_k,$$

where $o(p) = v_1$, $t(p) = v_2$, every p_i is a non-trivial ST_m -path, every s_i is a path in a C_i -component Q_i of Γ , and Q_0 is attached to p_1 , Q_i is attached both to p_i^{-1} and p_{i+1} for $i \in [1, k-1]$, and Q_k is attached to p_k^{-1} . It is easy to see that every ST_m -path is an ST_m -subgraph which does not contain non-trivial \mathcal{C}_m -components. Moreover, if Γ is freely folded then every ST_m -subgraph without non-trivial \mathcal{C}_m -components is an ST_m -path.

A maximal (by inclusion) ST_m -subgraph we call an ST_m -complex.

Let Q_1 be a C_1 -component and Q_2 a C_2 -component of an ST_m -complex P . Let p be any path from Q_1 to Q_2 in P . For any loop q at $t(p)$ in Q_2 we have $c_1 = \overline{\mu(p)} * \overline{\mu(q)} * \overline{\mu(p)}^{-1} \in C_1$ but it may happen that $c_1 \notin H_{C_1}(Q_1)$. In other words, at a vertex of Q_1 there may be a loop in P whose reduced label belongs to C_1 but is not readable in Q_1 . In this case we can enlarge $H_{C_1}(Q_1)$ by adding a loop at $o(p)$ labeled by c_1 . More precisely, let $q = e_1 \cdots e_k$, where $\mu(e_i) = u_i^{n_i}$, $u_i \in U_m \cap C_2$ for each $i \in [1, k]$. We form a loop $q' = f_1 \cdots f_k$ such that $\mu(f_i) = v_i^{n_i}$, $v_i \in U_m \cap C_2$, where $v_i = \overline{\mu(q)} * u_i * \overline{\mu(q)}^{-1}$ for $i \in [1, k]$, and attach it to $o(p)$. Such an operation we call an *elementary loop translation*. That is, we translated a loop q from Q_2 to Q_1 . It is easy to see that an elementary loop translation is a graph morphism since the resulting graph is the original one together with a loop attached to one of its vertices.

A loop translation can be performed even if Q_1 is a trivial \mathcal{C}_m -component of P , that is, Q_1 is a single vertex.

Lemma 19. *Let Γ and Δ be $\mathcal{B}(G)$ -graphs such that Δ is obtained from Γ by an elementary loop translation ϕ in an ST_m -complex P of Γ . If $v_0, v \in V(\Gamma)$ and $v'_0 = \phi(v_0)$, $v' = \phi(v)$ then*

$$L(\Gamma, v_0, v) = L(\Delta, v'_0, v').$$

Proof. We prove this lemma for the case $v = v_0$. The argument can be easily adjusted to accommodate the case $v \neq v_0$.

Let p be a path from Q_1 to Q_2 in Γ , where Q_1 is a C_1 -component and Q_2 a C_2 -component of the ST_m -complex P , and Δ is obtained from Γ by translation of a loop q from $t(p) \in V(Q_2)$ to $o(p) \in V(Q_1)$. The new loop in Q_1 we denote by q' . With abuse of notation we denote $\phi(p)$ and $\phi(q)$ again respectively by p and q in Δ .

Obviously $L(\Gamma, v) \subseteq L(\Delta, v')$ since Γ is graph-isomorphic to $\phi(\Gamma)$.

Now let r be a reduced loop at v' in Δ . If r does not contain q' then $\phi^{-1}(r)$ is a loop at v in Γ and $\mu(r) = \mu(\phi^{-1}(r))$, that is, $\overline{\mu(r)} \in L(\Delta, v')$. Now, suppose r contains loops q' , so we subdivide it as follows

$$r = r_1 q' r_2 \cdots r_{k-1} q' r_k,$$

where $t(r_i) = o(r_{i+1}) = o(p)$, $i \in [1, k-1]$. Hence, $r' = r_1 r_2 \cdots r_k$ is a loop at v' which does not contain q' and hence there exists its preimage $\phi^{-1}(r') = \phi^{-1}(r_1) \cdots \phi^{-1}(r_k)$ in Γ which is a loop at v . Now, for each entry of q' in r we insert a loop $p q p^{-1}$ in P into $\phi^{-1}(r')$, that is, we consider

$$r'' = \phi^{-1}(r_1) (p q p^{-1}) \phi^{-1}(r_2) \cdots \phi^{-1}(r_{k-1}) (p q p^{-1}) \phi^{-1}(r_k).$$

Since $\overline{\mu(q')} = \overline{p q p^{-1}}$ and $\overline{\mu(r_1)} = \overline{\mu(\phi^{-1}(r_1))}$ then $\overline{\mu(r)} = \overline{\mu(r'')} \in L(\Gamma, v)$. That is, $L(\Delta, v') \subseteq L(\Gamma, v)$. □

Since $H_{C_2}(Q_2)$ is finitely generated then it is enough to perform only finitely many elementary loop translations to obtain

$$\overline{\mu(p)} * H_{C_2}(Q_2) * \overline{\mu(p)}^{-1} \subseteq H_{C_1}(Q_1)$$

in the resulting graph.

If an ST_m -complex P contains finitely many C_m -components (including trivial ones) then after application of finitely many elementary loop translations we get

$$\overline{\mu(p_{i,j})} * H_{C_j}(Q_j) * \overline{\mu(p_{i,j})}^{-1} = H_{C_i}(Q_i),$$

where C_i, C_j are (possibly trivial) C_m -components of P and $p_{i,j}$ is a path from Q_i to Q_j in P . In this event we call P a *balanced* ST_m -complex.

We call an ST_m -complex P *reduced* if for every C -component Q of P , $C \in \mathcal{C}_m$, there are no two distinct edges $e_1, e_2 \in E(P)$ such that $o(e_1), o(e_2) \in V(Q)$, $\mu(e_1) = \mu(e_2) = t \in T_m^{\pm 1}$. Observe that the condition $e_1, e_2 \in E(P)$ implies $C = C_t$. It is easy to see that the property “to be reduced” for an ST_m -complex is similar to the property “to be freely folded” for a $\mathcal{B}(G)$ -graph defined above. The latter property can be obtained by a sequence of free foldings, and now we introduce a

similar operation, application of which transforms an arbitrary ST_m -complex into a reduced one.

Let Q be a C -component of an ST_m -complex P and suppose there exist distinct edges e, f such that $o(e), o(f) \in V(Q)$, $\mu(e) = \mu(f) = t \in T_m^{\pm 1}$. Let $p = e_1 \cdots e_k$ be a path in Q such that $o(p) = o(e)$, $t(p) = o(f)$, $\mu(e_i) = u_i^{n_i}$, where $u_i \in U_m \cap C$ is a power of a generator of C for each $i \in [1, k]$. Observe that $C_t = C$. Now, we form a path $q = f_1 \cdots f_k$ such that $\mu(f_i) = v_i^{n_i}$, $v_i \in U_m \cap D_t$, where $v_i = t^{-1} * u_i * t$ for $i \in [1, k]$. Finally, we remove the edge e and attach q to Γ so that $o(q) = t(e)$, $t(q) = t(f)$. The resulting graph we denote by Δ and say that Δ is obtained from Γ by an ST_m -folding.

Lemma 20. *Let Γ and Δ be $\mathcal{B}(G)$ -graphs such that Δ is obtained from Γ by an ST_m -folding in an ST_m -complex P of Γ . If $v_0, v \in V(\Gamma)$ and $v'_0, v' \in V(\Delta)$ correspond to v_0, v respectively, then*

$$L(\Gamma, v_0, v) = L(\Delta, v'_0, v').$$

Proof. We prove this lemma for the case $v = v_0$. The argument can be easily changed to accommodate the case $v \neq v_0$.

We are going to use the notation from the definition of an ST_m -folding. That is, suppose there exist distinct edges e, f such that $o(e), o(f) \in V(Q)$, $\mu(e) = \mu(f) = t \in T_m^{\pm 1}$, where Q is a C -component of an ST_m -complex P of Γ . Suppose p is a path in Q from $o(e)$ to $o(f)$, whose edges are labeled by powers of generators of $C_t = C$, q is a corresponding path whose edges are labeled by powers of generators of D_t , and Δ is obtained from Γ by deleting e and attaching q .

With abuse of notation we are going to denote edges and vertices of Γ and Δ which correspond to each other by the same letters.

Let r be a reduced loop at v in Γ . We subdivide r as follows

$$r = p_1 e^{\delta_1} p_2 \cdots p_{k-1} e^{\delta_{k-1}} p_k,$$

where $o(p_1) = t(p_k) = v$ and $\delta_i \in \{-1, 1\}$, $i \in [1, k-1]$. Since in Δ the path pfq^{-1} begins at $o(e)$ and ends at $t(e)$ then we can consider a loop r' at v' in Δ , where

$$r' = p_1 (pfq^{-1})^{\delta_1} p_2 \cdots p_{k-1} (pfq^{-1})^{\delta_{k-1}} p_k.$$

Now, since $\overline{\mu(pfq^{-1})} = \mu(e)$, we have $\overline{\mu(r)} = \overline{\mu(r')}$ and $L(\Gamma, v) \subseteq L(\Delta, v')$.

Now, let r be a reduced loop at v' in Δ . Observe that since q is connected to Δ only at the endpoints and r is reduced it follows that if r contains an edge (or its inverse) which belongs to q then r contains

the whole path q . Hence, we can subdivide r as follows

$$r = p_1 q^{\delta_1} p_2 \cdots p_{k-1} e^{\delta_{k-1}} p_k,$$

where $o(p_1) = t(p_k) = v$ and $\delta_i \in \{-1, 1\}$, $i \in [1, k-1]$. Observe that the path $e^{-1}pf$ begins at $o(q)$ and ends at $t(q)$, so we can consider a loop r' at v in Γ , where

$$r' = p_1 (e^{-1}pf)^{\delta_1} p_2 \cdots p_{k-1} (e^{-1}pf)^{\delta_{k-1}} p_k.$$

Now, since $\overline{\mu(e^{-1}pf)} = \mu(q)$, we have $\overline{\mu(r)} = \overline{\mu(r')}$ and $L(\Delta, v') \subseteq L(\Gamma, v)$. □

Remark 3. Note that while ST_m -folding is not, strictly speaking, always a graph morphism, it is a *generalized morphism of $\mathcal{B}(G)$ -graphs*, that is a map $\theta : \Gamma_1 \rightarrow \Gamma_2$ between $\mathcal{B}(G)$ -graphs Γ_1, Γ_2 such that θ sends vertices to vertices, directed edges to directed paths, preserves labels of directed edges in the sense that $\overline{\mu(\theta(e))} = \overline{\mu(e)}$, and has the property that $o(\theta(e)) = \theta(o(e))$, $t(\theta(e)) = \theta(t(e))$ for any edge e of Γ_1 .

If P is a finite ST_m -complex of a $\mathcal{B}(G)$ -graph Γ then there exists a finite sequence of ST_m -foldings transforming P into a reduced ST_m -complex. Indeed, an ST_m -folding reduces the number of edges of P labeled by $T_m^{\pm 1}$, so the process stops in finitely many steps.

Next, due to the properties of T_m , for every $C \in \mathcal{C}_m$ there are at most two elements $t, s \in T_m$ attached to C . Hence, from the definition of ST_m -complex it follows that if Q is a C -component of P then it can be attached to edges labeled at most by two letters from $T_m^{\pm 1}$. So, in the corresponding reduced ST_m -complex there can be at most two edges outgoing from Q . Thus, if we contract all \mathcal{C}_m -components of P into vertices then the resulting graph $P / \sim_{\mathcal{C}_m}$ is a simple path (loop). This is summarized in the following lemma.

Lemma 21. *Let P be a balanced reduced ST_m -complex of a finite $\mathcal{B}(G)$ -graph Γ . Then P is a simple path (loop) modulo \mathcal{C}_m -equivalence. That is, every \mathcal{C}_m -component of P is attached to at most two edges labeled by $T_m^{\pm 1}$. Moreover, every vertex of P belongs to some \mathcal{C}_m -component of P and the subgroups associated to all \mathcal{C}_m -components are isomorphic.*

Proof. Follows from the discussion above. □

Lemma 22. *Let P be a balanced reduced ST_m -complex of a finite $\mathcal{B}(G)$ -graph Γ . If $v \in V(P)$, $w \in ST_m$, $u \in C_w$ then there exists an algorithm which effectively determines if there exists a path p in P such that $o(p) = v$ and $\overline{\mu(p)} = u * w$.*

Proof. If such a path p exists, the element w uniquely determines the sequence of ST_m -edges e_1, e_2, \dots, e_l that p traverses and, since P is reduced, which C -component Q of P contains the end-point $t(p)$. Let v_1, v_2, \dots, v_k be vertices of Q . For each v_i , fix an arbitrary ST_m -path q_i that originates at v , traverses ST_m -edges e_1, e_2, \dots, e_l and ends at v_i . Check whether $u * w \in \overline{\mu(q_i)} H_C(Q)$. Since P is balanced, if the answer is negative for every $i = 1, 2, \dots, k$ then there is no such p . If the answer is positive for some i_0 , then we can take $p = q_{i_0} q$, where q is a loop in Q such that $u * w = \overline{\mu(q_{i_0}) \mu(q)}$. \square

4. FOLDED $\mathcal{B}(G)$ -GRAPHS

As we noticed in Section 2, an element $g \in G$ can be represented by its normal form. Ideally, for a $\mathcal{B}(G)$ -graph Γ and its vertex v we would like to reduce the question “if $g \in L(\Gamma, v)$ ” to the question “if there exists a loop p in Γ at v labeled by $\pi(g)$ ”. The idea is that it would be much easier to answer the latter question than the former one, provided we could construct a completion of Γ which recognizes the same language but for which the above questions are equivalent.

However, certain technical problems arise which makes such a completion graph quite complicated. To avoid these technical difficulties, we impose a weaker condition on the graph which allows to keep its structure relatively simple, albeit at the cost of making reading procedure rather involved. To produce precise statements, we need to introduce some additional terminology first.

Recall that by Theorem 7,

$$G_i = \langle G_{i-1}, T_{i-1} \mid t^{-1} C_t t \stackrel{\phi_t}{=} D_t, \ t \in T_{i-1}, \ C_t, D_t \in \mathcal{C}_{i-1} \rangle.$$

Definition 10. Let Γ be a finite $\mathcal{B}(G)$ -graph. Then $\Gamma(i)$ is a subgraph of Γ such that if $e \in E(\Gamma(i))$ then either

- (1) $\mu(e) = x \in X^\pm$, or
- (2) $\mu(e) \in T_j$, $j < i$, or
- (3) $\mu(e) = v$, $v \in C \in \mathcal{C}_j$, $j < i$.

We call $\Gamma(i)$ the i -level graph of Γ (for example, the 1-level graph is the subgraph of Γ which consists only of edges with labels from X) and say that Γ has level N denoted by $l(\Gamma)$ if N is the minimal natural number for which $\Gamma = \Gamma(N)$.

Observe that $\Gamma(i)$ may not be connected for some $i < l(\Gamma)$, but all notions introduced in Section 3 still apply.

As we mentioned in Remark 2, normal forms are not unique. With that in mind we introduce a weaker version of standard decomposition for elements of G that still relies on the normal form theorem for HNN-extensions.

Definition 11. *To each element $g \in G$ we associate an (infinite) set of words $\Pi(g)$ as follows. If $g \in G_1$, then $\Pi(g) = \{\pi(g)\}$. Otherwise, assuming*

$$\pi(g) = \pi(g_1)w_1(r_1) \cdots \pi(g_k)w_k(r_k)\pi(g_{k+1}) = \pi(g_1)w_1(r_1)\pi(\bar{g}),$$

denote $C = C_{w_1}$, $D = D_{w_1}$, $\phi = \phi_{w_1}$ and put

$$\Pi(g) = \bigcup_{\substack{u \in C, v' \in D \\ v = \phi(u)}} \Pi(g_1 * u)(w_1 v') \Pi(v_{w_1}^{r_1} * v^{-1} * v'^{-1} * \bar{g}).$$

This definition does not depend on a particular choice of $\pi(g)$ by Lemma 9.

4.1. Weakly folded $\mathcal{B}(G)$ -graphs. In this subsection we introduce the conditions mentioned above and investigate the properties of $\mathcal{B}(G)$ -graphs that accommodate these conditions.

Definition 12. *Let $C \in \mathcal{C}_j$. A C -component Q is called label-maximal, if for any path p in Γ with $o(p) \in V(Q)$ and $\overline{\mu(p)} \in C$ there exists a path q in Q with $o(q) = o(p)$, $t(q) \in V(Q)$ and $\overline{\mu(q)} = \overline{\mu(p)}$.*

Let Q be a C -component for $C \in \mathcal{C}_j$. We say that the paths in Q are doubled if for a path q in Q labeled by $u \in C$ there exists a path p in Γ with $o(p) = o(q)$, $t(p) = t(q)$, $\mu(p) \in \Pi(u)$.

Definition 13. *We call a $\mathcal{B}(G)$ -graph Γ of level $N + 1$, $N \geq 0$ weakly folded if it has the following properties:*

- (P1) $\Gamma(N)$ is weakly folded,
- (P2) Γ is freely folded,
- (P3) each C -component of Γ is reduced ($C \in \mathcal{C}_N$),
- (P4) paths in all C -components are doubled ($C \in \mathcal{C}_N$),
- (P5) each C -component of Γ is label-maximal ($C \in \mathcal{C}_N$),
- (P6) each ST_N -complex of Γ is reduced and balanced.

In particular, a graph of level 1 is weakly folded if it is freely folded.

The following theorem provides statement of conditions and properties mentioned in the beginning of this section.

Theorem 23. *The following statements hold for each integer $N \geq 1$.*

- R**(N) *Let Γ be a weakly folded $\mathcal{B}(G)$ -graph of level N . Let $v_1, v_2 \in V(\Gamma)$. Let $g \in G$ be given as a word in the generators. Then there is an algorithm to check whether there exists a path p in Γ with $o(p) = v_1, t(p) = v_2, \mu(p) \in \Pi(g)$.*
- E**(N) *Let Γ be a weakly folded $\mathcal{B}(G)$ -graph of level N . Let $g \in G$ and let p be a path in Γ such that $\mu(p) = g$. Then there exists a path q in Γ with $o(q) = o(p), t(q) = t(p)$ and $\mu(p) \in \Pi(g)$.*
- C**(N) *Let Γ be a $\mathcal{B}(G)$ -graph of level N . Let $v_1, v_2 \in V(\Gamma)$. There is an algorithm to construct a weakly folded $\mathcal{B}(G)$ -graph Γ' and a generalized morphism $\Gamma \rightarrow \Gamma', v_1 \mapsto v'_1, v_2 \mapsto v'_2$ with $L(\Gamma, v_1, v_2) = L(\Gamma', v'_1, v'_2)$.*

We prove Theorem 23 by induction on N . The statements **R**(1), **E**(1), **C**(1) are about Stallings graphs in free groups and, therefore, they are clear. For $N > 1$, the induction steps for **R**(N), **E**(N), **C**(N) are shown in the following subsections. We will also establish a slightly stronger version of **R**(N) (coinciding with **R**(N) for $N = 1$).

- R**⁺(N) *Let Γ be a weakly folded $\mathcal{B}(G)$ -graph of level N . Let $v_1 \in V(\Gamma)$. Let $C \in \mathcal{C}_{N-1}$ and $g \in G_N$ be given as a word in the generators. Then there is an algorithm to check whether there exists an element of G of the form $gu, u \in C$ which is readable from v_1 in Γ . Moreover, there is an algorithm to produce a finite list of elements gu_1, gu_2, \dots , where $u_i \in C$, and a C -component Q in Γ such that each gu_i is readable from v_1 in Γ along a path terminating at a vertex of Q , and, further, any element of the form $gu, u \in C$ readable from v_1 in Γ belongs to the coset $(gu_i)H_C(Q)$.*

With a slight abuse of notation, we also call the effective procedures whose existence is provided in the statements **R**(N), **R**⁺(N), **C**(N) by the same letters **R**(N), **R**⁺(N), **C**(N) respectively.

The induction steps will be arranged as follows:

$$\begin{aligned}
\cdots &\Rightarrow \mathbf{E}(N) \Rightarrow \mathbf{E}(N+1) \Rightarrow \cdots, \\
\cdots &\Rightarrow \mathbf{R}(N), \mathbf{E}(N), \mathbf{C}(N), \mathbf{R}^+(N) \Rightarrow \\
&\Rightarrow \mathbf{C}(N+1) \Rightarrow \mathbf{R}(N+1), \mathbf{R}^+(N+1) \Rightarrow \cdots.
\end{aligned}$$

We would like to note that dependence of the reading procedure **R**($N+1$) on the folding procedure **C**($N+1$) at the same level is not strictly necessary, as **R**($N+1$) can be also deduced from **R**⁺(N). However, neither way provides an advantage over the other one in terms of computational complexity, so for the sake of making shorter argument, we choose the former way.

4.2. Induction step for $\mathbf{E}(\cdot)$. Suppose that the statement $\mathbf{E}(N)$ holds. In this subsection we prove $\mathbf{E}(N + 1)$.

Let q be a path in a weakly folded graph Γ of level $N + 1$, and $\overline{\mu(q)} = g$. Our goal is to show that there is also a path p from $o(q)$ to $t(q)$ that reads an element from $\Pi(g)$.

Let

$$\pi(g) = \pi(g_1)w_1(r_1) \cdots \pi(g_k)w_k(r_k)\pi(g_{k+1}).$$

At first we show that we can assume

$$\mu(q) = h_1(w_1v_1)h_2 \cdots h_k(w_kv_k)h_{k+1},$$

where $v_i \in D_{w_i}$ for $i \in [1, k]$. Indeed, by (P4) we can assume without loss of generality that

$$q = d_1e_1d_2 \cdots d_le_ld_{l+1},$$

where all d_i are paths in $\Gamma(N)$ and e_i are maximal (with respect to inclusion) ST_N -paths of maximal height. Denoting $t_i = \mu(e_i)$ and $\mu(d_i) = h'_i, \overline{h'_i} \in G_N$ we have

$$\mu(q) = h'_1t_1h'_2t_2 \cdots h'_lt_lh'_{l+1}.$$

If for every $i \in [1, l - 1]$ we have $D_{t_i} \neq C_{t_{i+1}}$, or $d_{i+1} \notin D_{t_i}$, then by Lemma 9, $k = l$ and $w_it_i^{-1} \in C_{w_i} = C_{t_i}$ for $i \in [1, l]$. Suppose $l \neq k$, therefore, there exists i such that $D_{t_i} = C_{t_{i+1}}$ and $d_{i+1} \in D_{t_i}$. Then by (P5) we can assume that d_{i+1} is a path inside of the D_{t_i} -component of $t(t_i)$, and $t_id_{i+1}t_{i+1}$ can be replaced either by a path in C_{t_i} -component of $o(t_i)$ (if $C_{t_i} = D_{t_{i+1}}$), or by an ST_N -path (otherwise). After such a replacement, the number l is reduced either by 2, or by 1, respectively.

Repeating this argument, we ultimately force $l = k$ and $w_it_i^{-1} \in C_{w_i} = C_{t_i}$ for $i \in [1, k]$.

We proceed by induction on k . If $k = 0$, by (P4) we assume that q is a path in $\Gamma(N)$ and by (P1) and $\mathbf{E}(N)$ there exists a path p in $\Gamma(N)$ such that $\mu(p) \in \Pi(g)$.

If $k > 0$ then we assume that we have established the statement for elements with less than k syllables and denote the piece of q corresponding to h_1 by q_1 . By (P4), we assume q_1 to be a path in $\Gamma(N)$, which, together with (P1), enables us to use $\mathbf{E}(N)$. Therefore, we may assume that $\mu(q_1) \in \Pi(h_1)$. Now reading $\Pi(g)$ is reduced to reading $\Pi(h_2w_2 \cdots w_kh_{k+1})$, which has at most $k - 1$ syllables.

Remark 4. Let $\pi(g)$ be as described in the beginning of the subsection. Suppose some path p reads $\pi'(g) \in \Pi(g)$. Then, given $l \in [1, k]$, we cannot guarantee that $g_1 \circ w_1(r_1) \circ \cdots \circ g_l$ is readable along an initial

subpath of p . Nevertheless, a weaker but still useful statement can be made: p can be subdivided into subpaths $p = p_1 p_2 p_3$ so that

$$\begin{aligned}\overline{\mu(p_1)} &= g_1 * w_1(r_1) * \cdots * g_l * u_l, \\ \overline{\mu(p_2)} &= u'_l * w_l * v'_l, \\ \overline{\mu(p_3)} &= v_l * g_{l+1} * \cdots * w_k(r_k) * g_{k+1},\end{aligned}$$

where $u_l, u'_l \in C_{w_l}$, $v_l, v'_l \in D_{w_l}$ and p_2 is an ST_N -path.

4.3. Induction step for $\mathbf{C}(\cdot)$. Now we present a procedure that, given a $\mathcal{B}(G)$ -graph Γ , allows to obtain a weakly folded graph that recognizes the same language. We proceed by induction on the level of Γ , that is, we assume $\mathbf{R}(N), \mathbf{R}^+(N), \mathbf{E}(N), \mathbf{C}(N)$ to hold, and prove $\mathbf{C}(N+1)$.

Let Γ be a graph of level $N+1$. By $\mathbf{C}(N)$ we can assume that $\Gamma(N)$ is weakly folded, thus gaining (P1). Note also that for graphs of level greater than 1 (P2) follows from (P1), (P3) and (P6).

Throughout this section, when we write $L(\Gamma) = L(\Gamma')$ we imply existence of a generalized morphism $\Gamma \rightarrow \Gamma'$ and we mean the equality $L(\Gamma, v_1, v_2) = L(\Gamma', v'_1, v'_2)$ for all $v_1, v_2 \in V(\Gamma)$ and $v_1 \mapsto v'_1, v_2 \mapsto v'_2$.

Lemma 24. *Let Γ be a finite $\mathcal{B}(G)$ -graph of level $N+1$ with the property (P1). Then using free foldings, simple reductions and the algorithm $\mathbf{C}(N)$ finitely many times, we can obtain a graph Γ' with $L(\Gamma) = L(\Gamma')$ for which (P1) and (P3) hold.*

Proof. Using finitely many simple reductions we can produce a graph Γ_1 , where all C -components of Γ ($C \in \mathcal{C}_N$) are reduced. Note that $\Gamma_1(N)$ may not be weakly folded anymore. This can only happen if some vertices in a C -component were identified, that is, if

$$\sum_Q |V(Q)| > \sum_{Q_1} |V(Q_1)|,$$

where Q_1 is a C -component of Γ_1 obtained from Q . By the induction hypothesis we can transform Γ_1 into a graph Γ_2 such that $\Gamma_2(N)$ is weakly folded. Let components Q_1 transform into Q_2 . Since edges labeled by $u \in C$ of non-maximal height belong to weakly folded $\Gamma_2(N)$, the only way Q_2 may cease to be reduced is when $\text{proj}_{u_{\max}} H_C(Q_1)$ gets changed, where u_{\max} is the generator of C of maximal height and $\text{proj}_{u_{\max}}$ stands for the projection onto the cyclic group $\langle u_{\max} \rangle$. So, one of components Q_2 may be not reduced only if two vertices in (coinciding or distinct) C -components Q_1 have been identified while transforming Γ_1 into Γ_2 , which increases $H_C(Q_1)$. Since finitely generated free abelian groups are noetherian, this can only happen finitely many times.

Therefore, in a finite number of steps we obtain a graph Γ' for which (P1) and (P3) hold. $L(\Gamma) = L(\Gamma')$ by Lemma 16 and Lemma 18. \square

Remark 5. *Note that strictly speaking, the algorithmic step in Lemma 24 is redundant since (P3) follows from (P1) and (P4). However, in certain cases obtaining (P3) prior to (P4) provides lower computational complexity.*

Lemma 25. *Let Γ be a finite $\mathcal{B}(G)$ -graph of level $N+1$ with the properties (P1) and (P3). Then adding finitely many vertices and edges, and using free foldings, simple reductions and the algorithm $\mathbf{C}(N)$ finitely many times, we can obtain a graph Γ' with $L(\Gamma) = L(\Gamma')$ for which (P1), (P3) and (P4) hold.*

Proof Let Q be a C -component for $C \in \mathcal{C}_N$. By $\mathbf{R}(N)$, for each edge $e \in Q$ check whether an element of $\Pi(\mu(e))$ is readable from $o(e)$ to $t(e)$ in $\Gamma(N)$. In the case it is not readable (which, by (P1), implies that $\mu(e)$ is of maximal height in C), add a path q from $o(e)$ to $t(e)$ with $\mu(q) \in \Pi(\mu(e))$. Note that this does not change the language defined by the graph Γ .

Thus, having added finitely many edges and vertices we provide that all paths are doubled in $\Gamma(N)$. This may have broken (P1). Using Lemma 24, obtain (P1) and (P3). Since the procedure in Lemma 24 does not produce new edges labeled by elements of maximal height in C , the property (P4) still holds. \square

In order to obtain (P5), we distinguish two cases: when the rank of C is equal to 1 and when the rank of C is bigger than 1, where C is one of the conjugated centralizers $C \in \mathcal{C}_N$. The following lemma is a technical statement we use to treat the former case.

Lemma 26. *Let Δ be a weakly folded $\mathcal{B}(G)$ -graph of level N , $v \in V(\Delta)$. Let $g \in G_N$ and paths p_i , $i = 1, 2, \dots$ be such that $o(p_i) = v$, $\overline{\mu(p_i)} = v_i * g * u_i$, where $v_i \in C \in \mathcal{C}_{\leq N-1}$, $u_i \in D \in \mathcal{C}_{\leq N-1}$. Then*

- (a) *if $[g^{-1}Cg, D] \neq 1$ then there exist i_0, i_1 such that $v_{i_0} * v_{i_1}^{-1} \in H_C(v)$,*
- (b) *if $[g^{-1}Cg, D] = 1$ then there exists a subsequence $\{i_j\}$, $j = 1, 2, \dots$ of $\{i\}$ such that all p_{i_j} terminate at the same vertex v_t , and there exist loops l_j in the D' -component of v_t such that $\overline{\mu(p_{i_j} p_{i_{j+1}}^{-1} l_j)} = 1$, where D' is an abelian subgroup of G of maximal height such that $D \subseteq D' \in \mathcal{C}_{\leq N-1}$.*

Proof. (b) is obvious. Indeed, denoting $g^{-1} * v_i * g = u'_i$, we have $\overline{\mu(p_i)} = g * u'_i * u_i$, where $u_i, u'_i \in D'$. Then the subsequence i_j , such that p_{i_j} terminate at the same vertex, delivers the statement.

We prove (a) by reducing it to an analogous statement about G_{N-1} and shifts v'_i, u'_i that are elements of centralizers that belong to $\mathcal{C}_{<N-1}$.

Denote the generator of C of maximal height by v_{\max} and put $v_i = v_{\max}^{k_i} * v'_i$, where $v'_i \in C' \leq C$, $C' \in \mathcal{C}_{<N-1}$. In particular, $v_i = v'_i$ if $C \in \mathcal{C}_{<N-1}$. Similarly, denote the generator of D of maximal height by u_{\max} and put $u_i = u_{\max}^{m_i} * u'_i$, where $u'_i \in D' \leq D$, $D' \in \mathcal{C}_{<N-1}$. If the set $\{k_i \mid i = 1, 2, \dots\}$ is bounded, passing to a subsequence we may assume that $v_i = v_{\max}^k * v'_i$. Similarly, if the set $\{m_i \mid i = 1, 2, \dots\}$ is bounded, we may assume that $u_i = u_{\max}^m * u'_i$. Using Lemma 8, and, if necessary, passing to a subsequence, we conclude that all elements $v_i * g * u_i$ have one of the following four forms:

$$\begin{aligned} \text{Case 1 } & \begin{cases} v_i * g * u_i = v'_i * [v_{\max}^k * g * u_{\max}^m] * u'_i, \text{ or} \\ v_i * g * u_i = v'_i * [v_{\max}^k * g * u_{\max}^M] \circ u_{\max}^{m_i-M} * u'_i, \end{cases} \\ & \text{or} \\ \text{Case 2 } & \begin{cases} v_i * g * u_i = (v'_i * v_{\max}^{k_i-M}) \circ [v_{\max}^M * g * u_{\max}^m] * u'_i, \text{ or} \\ v_i * g * u_i = (v'_i * v_{\max}^{k_i-M}) \circ [v_{\max}^M * g * u_{\max}^M] \circ (u_{\max}^{m_i-M} * u'_i). \end{cases} \end{aligned}$$

Consider the cases above.

Case 1. For simplicity, assume the bracketed part to be equal to g . Let

$$\pi(g) = \pi(g_1) w_1(r_1) \pi(g_2) w_2(r_2) \cdots w_k(r_k) \pi(g_{k+1}),$$

where $w_i \in ST_{N-1}$.

Suppose $[g_1^{-1} C g_1, C_{w_1}] = 1$ and $k > 0$. Then, since $v'_i \in G_{N-1}$, by **E(N)** (see Remark 4), the paths p_i can be subdivided into subpaths q'_i $q_i = p_i$ that read

$$\overline{\mu(q'_i)} = v'_i * g_1 * w_1 * u''_i, \quad \overline{\mu(q_i)} = u''_i^{-1} * g_2 * w_2 * \cdots * g_{k+1} * u_i,$$

where $u''_i \in D_{w_1}$. Then, by (b) applied to q'_i , we pass to a subsequence of q_i 's and the word $g' = g_2 \circ w_2(r_2) \circ \cdots \circ g_{k+1}$ of smaller level N syllable length than g . Either g' has level N syllable length 0, or greater than 0 (that is, $g \in G_N - G_{N-1}$), and $u''_i \in G_{N-1}$.

Hence, repeating the argument, we eventually may assume that $k > 0$ and $[g_1^{-1} C g_1, C_{w_1}] \neq 1$, or $k = 0$ and, by the assumption of (a), $[g_1^{-1} C g_1, D] \neq 1$. By **E(N)** and Remark 4, there exist a collection of paths q_i in $\Delta(N-1)$ reading $\overline{\mu(q_i)} = v'_i * g_1 * u''_i$, where either $u''_i \in C_{w_1}$ if $k > 0$, or $u''_i \in D'' \in \mathcal{C}_{<N-1}$, $[D'', D] = 1$, if $k = 0$. The statement is thus reduced to graphs of lower level. Note that if the rank of C is

equal to 1 then this case delivers the statement of the lemma, since in this event $v'_i = 1$.

Case 2. Show further that in this case we still may assume (concatenating p_i with a loop at v) that the set $\{k_i - M \mid i = 1, 2, \dots\}$ is bounded. More precisely, show that the C -component Q of v contains a loop labeled by $v_{\max}^{k_0} * v'$, where $v' \in C$ is of non-maximal height.

Indeed, $\pi(v_i * g * u_i)$ begins with $\pi(v'_i * v_{\max}^{k_i - M})$, therefore, the loop in question exists by **E**(N) and **E**($N - 1$) since Δ contains only a finite number of C' -components.

Denote loop at v reading $v_{\max}^{k_0} * v'$ by q . Then with an appropriate choice of powers l_i we can make $q^{l_i} p_i$ satisfy **Case 1**. □

Lemma 27. *Let Δ be a weakly folded $\mathcal{B}(G)$ -graph of level N and let $v \in V(\Delta)$. Let $u \in G_N$ be such that $\pi(u^2) = \pi(u)^2$. Then there is an algorithm to construct the label-maximal $\langle u \rangle$ -component at v .*

Proof. In other words, we need to find all end vertices of paths p that read u^k starting from v .

Let $\pi(u) = \pi(h_1) w_1(r_1) \pi(h_2) w_2(r_2) \cdots w_k(r_k) \pi(h_{k+1})$. By **R**⁺(N), we consecutively attempt to read $\Pi(u)$, $\Pi(u) h_1 v'$, $\Pi(u^2)$, $\Pi(u^2 h_1) v'$, \dots , where $v' \in C = C_{w_1}$, starting from v in Δ . We stop if either

- (a) some $\Pi(u^N h_1) v'$ is not readable for any $v' \in C$, or
- (b) we discover that some $\Pi(u^N)$ is readable as a label of a loop at v .

If (a) occurs, by **E**(N) and Remark 4 no loop labeled by a power of u is readable at the vertex v , so it is enough to check which of $\Pi(u^{\pm 1}), \dots, \Pi(u^{\pm N})$ are readable as labels of paths from v .

If (b) occurs, then for any u^M readable starting from v , $u^{M \pm N}$ is also readable with the same terminal vertex, so it is enough to check which of $\Pi(u^{\pm 1}), \dots, \Pi(u^{\pm N})$ are readable as labels of paths from v .

Now we show that it takes only finitely many steps for either one of (a) and (b) to occur. Either at some point $\Pi(u^N h_1) v'$ is readable for no $v' \in C$, or endpoints of the corresponding paths meet some C -component Q arbitrarily many times.

The former case is exactly (a). In the latter case, show that there exists a loop at v reading a power of u .

Note that if paths p_i ($i = 1, 2, \dots$) are such that $o(p_i) = v$, $t(p_i) \in Q$, $\overline{\mu(p_i)} = u^{k_i} * h_1 * v_i$, then concatenating with a path in Q we may assume that $t(p_i) = t(p_j)$ for all i, j . Then by **E**(N), each p_i contains a terminal subpath q_i with $\overline{\mu(q_i)} = u_i * h_{k+1} * h_1 * v_i$, where $u_i \in D_{w_k}$. Since $\pi(u^2) = \pi(u)^2$, the paths q_i^{-1} satisfy conditions of Lemma 26

(a), therefore, for some loop q in Q we have $\overline{\mu(p_{i_0}qp_{i_1}^{-1})} = u^l$, where $l = k_{i_0} - k_{i_1}$. □

Remark 6. *The technical difficulty in solving power problem for u in $L(\Delta)$ by direct inspection of Δ is caused by the fact that there is no clear connection between $\Pi(u)$ and $\Pi(u^2)$, which makes it theoretically possible to have a weakly folded graph Δ , in which $\Pi(u^M)$ can be read as a label of a loop at a vertex v , but $\Pi(u^{<M})$ cannot be read as a label of a path from v . We would like to point out that this is essentially the same difficulty that one faces trying to solve the membership problem for \mathbb{Z}^n -free groups using the machinery of [8], as we mentioned in Section 2.4 on page 10.*

Lemma 28. *Let Δ be a weakly folded $\mathcal{B}(G)$ -graph of level N and let $v \in V(\Delta)$. Let $C \in \mathcal{C}_N$ be such that the rank of C is greater than 1. Then there is an algorithm to construct the label maximal C -component at v .*

Proof. In other words, we need to find all end vertices of paths whose labels are equal to $u = u_{\max}^k * u'$, where u_{\max} is a generator of C of maximal height and u' is an element of less than maximal height. To do that it is enough to inspect the ST_{N-1} -complex of v using Lemma 22. □

Lemma 29. *Let Γ be a finite $\mathcal{B}(G)$ -graph of level $N + 1$ with the properties (P1), (P3) and (P4). Then adding finitely many vertices and edges, using free foldings, simple reductions and the algorithm $\mathbf{C}(N)$ finitely many times, we can obtain a graph Γ' with $L(\Gamma) = L(\Gamma')$ for which (P1), (P3)–(P5) hold.*

Proof. To acquire (P5), we apply either Lemma 27, or Lemma 28 to all $C \in \mathcal{C}_N$ and all $v \in V(\Gamma(N))$. Note that since $\Gamma(N)$ is weakly folded this procedure does not break (P1), (P3) and (P4). □

Lemma 30. *Let Γ be a finite $\mathcal{B}(G)$ -graph of level $N + 1$ with the properties (P1), (P3)–(P5). Then adding finitely many vertices and edges, using free foldings, simple reductions, loop translations, ST_N -foldings and the algorithm $\mathbf{C}(N)$ finitely many times we can obtain a graph Γ' with $L(\Gamma) = L(\Gamma')$ for which (P1)–(P6) hold.*

Proof. Let P be an ST_N -complex. As shown before, using finitely many loop translations and ST_N -foldings, we can transform Γ into a

graph Γ_1 such that the ST_N -complex P_1 corresponding to P is a reduced balanced complex. Then, using previous lemmas, we transform Γ_1 into a graph Γ_2 for which (P1), (P3)–(P5) hold. The ST_N -complex P_2 corresponding to P_1 may cease to be reduced in the only case: some C -components of Γ_1 ($C \in \mathcal{C}_N$) got glued together. Since the number of C -components is finite, repeating this procedure we may assume P_2 to be reduced. Then P_2 may cease to be balanced only if some C -components got changed in the process, increasing $H_C(Q)$. This can only happen a finite number of times, since $H_C(Q)$ is a subgroup of a noetherian group.

Eventually, we obtain a graph Γ' for which (P1), (P3)–(P6) (and therefore (P2)) hold. $L(\Gamma) = L(\Gamma')$ by Lemmas 19, 20 and 24–29. \square

Theorem 31. *Let Γ be a finite $\mathcal{B}(G)$ -graph of level $N+1$. Then adding finitely many vertices and edges, using free foldings, simple reductions, loop translations, ST_N -foldings and the algorithm $\mathbf{C}(N)$ finitely many times we can obtain a weakly folded graph Γ' with $L(\Gamma) = L(\Gamma')$.*

Proof. Immediate from Lemmas 24–30. \square

Note that all the operations listed in the theorem above are either graph morphisms, or generalized graph morphisms, so for each vertex v of Γ there is a corresponding vertex v' in Γ' .

4.4. Induction step for $\mathbf{R}(\cdot)$ and $\mathbf{R}^+(\cdot)$. Assume that the statements $\mathbf{R}(N)$, $\mathbf{E}(N)$, $\mathbf{C}(N+1)$ hold. In this subsection we prove $\mathbf{R}(N+1)$ and $\mathbf{R}^+(N+1)$.

Fix a weakly folded graph Γ of level $N+1$. Suppose $g \in G_{N+1}$. Build a graph Γ' of level $N+1$ by attaching a path p_g reading g to Γ at a vertex v_1 , that is, such that $o(p_g) = v_1$, $\mu(p_g) \in \Pi(g)$. Next, by $\mathbf{C}(N+1)$ construct a weakly folded graph Γ'' with $L(\Gamma'') = L(\Gamma')$. Denote by v'' the vertex in Γ'' that corresponds to $t(p_{g_1})$ in Γ' . The question whether $\Pi(g)$ is readable from v_1 to v_2 in Γ is now reduced to the question whether v'' corresponds to the vertex v_2 of Γ .

To establish $\mathbf{R}^+(N+1)$ it is enough to inspect the C -component ($C \in \mathcal{C}_N$) of v'' . More exactly, we note that Γ'' defines the same subgroup of G as Γ , so no vertices of (distinct or coinciding) C -components of Γ could be identified while obtaining Γ'' . Suppose a path gu , $u \in C$ is readable in Γ from v_1 . Then it is also readable in Γ'' along the path p_{gu} from the image of v_1 . By (P4), v'' and $t(p_{gu})$ belong to the same C -component Q'' of Γ'' . Denote by Q the C -component of Γ that corresponds to Q'' in Γ'' and let $V(Q)$ be the finite set of vertices of

Q . Denote by $V''(Q'')$ the set of vertices of Q'' that correspond to $V(Q)$. Denote by u_1, u_2, \dots labels of arbitrary paths inside Q'' from v'' to vertices in $V''(Q'')$. Then the elements $g * u_1, g * u_2, \dots$ and the C -component Q deliver the statement $\mathbf{R}^+(N + 1)$.

5. RESULTS AND CONCLUSIONS

Before giving some corollaries of Theorem 23 we discuss applicability of the obtained results.

5.1. Note on effectiveness. The main question that arises is the following. Suppose the input is a presentation of group a G and the fact that it acts freely and regularly on *some* \mathbb{Z}^n -tree. Can we solve the uniform subgroup membership problem in G using the techniques developed in the present work? Strictly speaking, the answer is unfortunately negative (at least, to the date). Indeed, Theorem 23 gives an effective construction of a weakly folded $\mathcal{B}(G)$ -graph and subsequently means to read words as labels of paths in the graph, thus solving the uniform membership problem for a fixed ambient G (see Theorem 32 below). More exactly, it proves existence of a decision algorithm but only gives a way to obtain such an algorithm if the data in Theorem 23 is given as a part of the input. However, if we are concerned with uniformity over different groups G , to build and use weakly folded graphs, we have to address certain issues.

To elaborate, suppose first that the chain of HNN-extensions for G as described in Theorem 5 is given as a part of the input (we say in this case that G is *given by an effective HNN-chain*). Can one then assume that the Lyndon length function of any element of G can be computed effectively? The answer is positive. Indeed, as one can see from the proof of Theorem 5 and Theorem 6 (see [10]), an effective HNN-chain provides an algorithm to embed G into a partial group of infinite words, and, moreover, the images of generators are recursive as functions defined on segments of \mathbb{Z}^n . Using standard diagonal argument, one can effectively find $\text{com}(f, g)$ if f, g are given as recursive \mathbb{Z}^n -words. Then it is easy to see that the embedding, the group operation and the Lyndon length function are effective.

Therefore, the question is reduced to whether one can algorithmically obtain an effective HNN-chain given a group presentation and information that the group acts freely and regularly on a \mathbb{Z}^n -tree. To the date, such an algorithm is not known. However, a number of researchers express belief that a Makanin–Razborov type machinery can be employed to give effective versions of Theorems 1, 3 and ultimately Theorem 5.

Note that similar questions arise even regarding solution of the Word Problem in G . Indeed, a simple solution that follows from Bass–Serre theory (presented, for example, in [14]), relies on constructing a graph of groups based on the group action on a \mathbb{Z}^n -tree. While this proves existence of a decision algorithm for a fixed G , it does not suggest any means to produce such an algorithm given a presentation of G and information that it acts freely on some \mathbb{Z}^n -tree.

Finally, we would like to point out that requiring effectiveness of the Lyndon length function (or of the embedding into infinite words) does not seem unreasonable. Indeed, it is hard to expect that information of existence of an action on a \mathbb{Z}^n -tree comes ad hoc, without robust relation to the presentation of G . This thought is reinforced by the fact that the most powerful to the date tool to produce examples of \mathbb{Z}^n -free groups, Theorem 6, implies effective embedding into infinite words, as mentioned above.

5.2. Solutions to the Subgroup Membership and Power Problems. The following theorem states that the uniform membership problem is decidable in \mathbb{Z}^n -free groups.

Theorem 32. *Let G be a finitely generated \mathbb{Z}^n -free group. There exists an algorithm that, given $g, h_1, h_2, \dots, h_k \in G$ presented as words in the generators of G , decides whether $g \in \langle h_1, h_2, \dots, h_k \rangle$.*

Proof. Represent G as a chain of HNN-extensions as described in Theorem 7. Find normal forms $\pi(g), \pi(h_1), \pi(h_2), \dots, \pi(h_k)$. Consider the graph Γ_0 which is a bouquet of k loops q_1, \dots, q_k labeled by $\mu(q_i) = \pi(h_i)$, $i \in [1, k]$. Apply **C**(N) (Theorem 23) for $N = K(G)$ to build a weakly folded version of Γ_0 , denoted Γ . Using **R**(N) (Theorem 23) decide whether an element of $\Pi(g)$ is readable as a label of a loop in Γ . By **E**(N) (Theorem 23), the positive answer implies $g \in \langle h_1, h_2, \dots, h_k \rangle$, while the negative answer implies $g \notin \langle h_1, h_2, \dots, h_k \rangle$. \square

The following theorem states that the power problem is decidable in \mathbb{Z}^n -free groups.

Theorem 33. *Let G be a finitely generated \mathbb{Z}^n -free group. There exists an algorithm that, given $g, h_1, h_2, \dots, h_k \in G$ presented as words in generators of G , decides whether there exists an integer $l \neq 0$ such that $g^l \in \langle h_1, h_2, \dots, h_k \rangle$.*

Proof. Represent G as a chain of HNN-extensions as described in Theorem 7. Find $c = \text{com}(g, g^{-1}) \in G$, so that $g_0 = c^{-1} * g * c$ is cyclically reduced. Then by Lemma 12 we can find $d \in G$ such that

$g_1 = d^{-1} * g_0 * d = d^{-1} * c^{-1} * g * c * d$ is either cyclically normal, or $g_1 = u * w$, $u \in C_w = D_w$.

Consider the graph Γ_0 that is a bouquet of k loops q_1, \dots, q_k labeled by $\mu(q_i) = \pi(h_i)$, $i \in [1, k]$ and a path q attached to origin of the bouquet such that $\mu(q) = \pi(c * d)$. By $\mathbf{C}(N)$ (Theorem 23) build a weakly folded version of Γ_0 , denoted Γ . Then using either Lemma 27 (in the former case), or Lemma 28 (in the latter case) check whether a loop labeled by a power of g_1 is readable from $t(q)$ in Γ . The positive answer implies that

$$g_1^l \in \langle d^{-1}c^{-1}h_1cd, d^{-1}c^{-1}h_2cd, \dots, d^{-1}c^{-1}h_kcd \rangle,$$

which equivalent to $g^l \in \langle h_1, h_2, \dots, h_k \rangle$. The negative answer implies that there is no l such that $g^l \in \langle h_1, h_2, \dots, h_k \rangle$. □

As discussed in Section 5.1, Theorem 32 and Theorem 33 state existence of decision algorithms but do not provide means to find such algorithms. The following theorem provides a way to make the group G a part of the input, although at the cost of making an extra assumption on G .

Theorem 34. *There exists an algorithm that given*

- (1) *a group G as an effective HNN-chain,*
- (2) *elements $g, h_1, h_2, \dots, h_k \in G$ presented as words in the generators of G ,*

decides whether $g \in \langle h_1, h_2, \dots, h_k \rangle$ and whether there exists an integer $l \neq 0$ such that $g^l \in \langle h_1, h_2, \dots, h_k \rangle$.

Proof. Follows from the discussion in Section 5.1 and Theorems 32 and 33. □

REFERENCES

- [1] H. Bass. Groups acting on non-arhimedean trees. In *Arboreal group theory, MSRI Publications*, volume 19, pages 69–130. New York: Springer-Verlag, 1991.
- [2] I. Chiswell. *Introduction to Λ -trees*. World Scientific, 2001.
- [3] I. Chiswell. A -free groups and tree-free groups. *Contemp. Math., Amer. Math. Soc.*, 378:79–86, 2005.
- [4] I. Chiswell and T. Muller. Embedding theorems for tree-free groups. Under consideration, 2010.
- [5] A. M. W. Glass. Partially ordered groups. *Series in Algebra*, 7, 1999.
- [6] V. Guirardel. Limit groups and groups acting freely on \mathbb{R}^n -trees. *Geom. Topol.*, 8:1427–1470, 2004.

- [7] I. Kapovich and A. Myasnikov. Stallings foldings and subgroups of free groups. *J. of Algebra*, 248:608–668, 2002.
- [8] I. Kapovich, A. Myasnikov, and R. Weidmann. Foldings, graphs of groups and the membership problem. *Int. J. of Algebra and Computation*, 15(1):95–128, 2005.
- [9] O. Kharlampovich, A. Myasnikov, V. Remeslennikov, and D. Serbin. Subgroups of fully residually free groups: algorithmic problems. In A. G. Myasnikov and V. Shpilrain, editors, *Group theory, Statistics and Cryptography*, volume 360, pages 63–101, 2004.
- [10] O. Kharlampovich, A. Myasnikov, V. Remeslennikov, and D. Serbin. Groups with free regular length functions in \mathbb{Z}^n . To appear in *Trans. Amer. Math. Soc.*, arXiv:0907.2356v2, 2010.
- [11] O. Kharlampovich, A. Myasnikov, and D. Serbin. Regular completions of \mathbb{Z}^n -free groups. Submitted, 2011.
- [12] V. Kopytov and N. Medvedev. *Right-ordered groups*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1996.
- [13] A. Martino and S. O Rourke. Some free actions on non-archimedean trees. *J. of Group Theory*, 7(2):275–286, 2004.
- [14] A. Martino and S. O Rourke. Free actions on \mathbb{Z}^n -trees: a survey. Preprint, 2005.
- [15] J. Morgan and P. Shalen. Valuations, trees, and degenerations of hyperbolic structures. *I. Annals of Math*, 120(3):401–476, 1984.
- [16] A. Myasnikov, V. Remeslennikov, and D. Serbin. Regular free length functions on Lyndon’s free $\mathbb{Z}[t]$ -group $F^{\mathbb{Z}[t]}$. *Contemp. Math., Amer. Math. Soc.*, 378:37–77, 2005.
- [17] A. Myasnikov, V. Remeslennikov, and D. Serbin. Fully residually free groups and graphs labeled by infinite words. *Internat. J. Algebra Comput.*, 16(4):689–737, 2006.
- [18] A. Nikolaev and D. Serbin. Finite index subgroups of fully residually free groups. *Internat. J. Algebra Comput.*, 21(4):651–673, 2011.
- [19] E. Rips. Subgroups of small cancellation groups. *Bull. London Math. Soc.*, 14:45–47, 1982.
- [20] J.-R. Stallings. Topology of finite graphs. *Invent. Math.*, 71:551–565, 1983.

STEVENS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICAL SCIENCES,, 1 CASTLE POINT ON HUDSON, HOBOKEN, NJ 07030